

CONTEMPORARY MATHEMATICS

624

The Mathematics of Decisions, Elections, and Games

AMS Special Sessions on
The Mathematics of Decisions, Elections, and Games
January 4, 2012, Boston, MA
January 11–12, 2013, San Diego, CA

Karl-Dieter Crisman
Michael A. Jones
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Contents

Preface	vii
Redistricting and District Compactness CARL CORCORAN and KAREN SAXE	1
Fair Division and Redistricting ZEPH LANDAU and FRANCIS EDWARD SU	17
When Does Approval Voting Make the “Right Choices”? STEVEN J. BRAMS and D. MARC KILGOUR	37
How Indeterminate is Sequential Majority Voting? A Judgement Aggregation Perspective KLAUS NEHRING and MARCUS PIVATO	55
Weighted Voting, Threshold Functions, and Zonotopes CATHERINE STENSON	89
The Borda Count, the Kemeny Rule, and the Permutahedron KARL-DIETER CRISMAN	101
Double-Interval Societies MARIA MARGARET KLAWE, KATHRYN L. NYMAN, JACOB N. SCOTT, and FRANCIS EDWARD SU	135
Voting for Committees in Agreeable Societies MATT DAVIS, MICHAEL E. ORRISON, and FRANCIS EDWARD SU	147
Selecting Diverse Committees with Candidates from Multiple Categories THOMAS C. RATLIFF	159
Expanding the Robinson-Goforth System for 2×2 Games BRIAN HOPKINS	177
Cooperation in n -Player Repeated Games DANIEL T. JESSIE and DONALD G. SAARI	189
The Dynamics of Consistent Bankruptcy Rules MICHAEL A. JONES and JENNIFER M. WILSON	207

Preface

The majority of papers in this collection accompanied talks from the AMS Special Sessions on the Mathematics of Decisions, Elections, and Games from the 2012 and 2013 Joint Mathematics Meetings of the AMS and MAA. These sessions were organized by Karl-Dieter Crisman (Gordon College), Michael A. Jones (Mathematical Reviews), and Michael Orrison (Harvey Mudd College). The one exception is the paper based on a talk from the AMS Special Session on The Redistricting Problem; this session was organized by Daniel Goroff (Harvey Mudd College) and Daniel Ullman (George Washington University) for the 2009 Joint Mathematics Meetings. The full programs for the 2012 and 2013 AMS Special Sessions on the Mathematics of Decisions, Elections, and Games can be found online by searching jointmathematicsmeetings.org.

Decision theory, voting theory, and game theory are three intertwined areas of mathematics that involve making optimal decisions under different contexts. Although these areas consist of their own mathematical results, much of the recent research in these areas involve developing and applying new perspectives from their intersection with other branches of mathematics, such as algebra, representation theory, combinatorics, convex geometry, dynamical systems, etc. The papers in this volume highlight and exploit the mathematical structure of decisions, elections, and games to model and to analyze problems from the social sciences.

In what follows we give a short overview of the papers in this collection. To those new to the area, we wish to emphasize that many different types of mathematics can be profitably used in this interdisciplinary context.

Both *Redistricting and District Compactness* by Corcoran and Saxe and *Fair Division and Redistricting* by Landau and Su focus on the redistricting problem: carving up a state into congressional districts. The former discusses different measures used to evaluate proposed districting plans. These measures take into account the geometric shape of both the state and the districts as well as the spread of the population throughout the district. The latter paper offers a different perspective on redistricting by viewing it as a fair division or cake-cutting problem in which the state is viewed as a cake to be divided into districts that are allocated to one of two political parties.

The next two papers fall under the topic of judgment aggregation, which may be viewed as being at the intersection of social choice theory and decision theory. While social choice theory is concerned with aggregating preferences of voters to arrive at a societal ranking, judgment aggregation is a burgeoning area focusing on aggregating individuals' judgments on interconnected propositions to arrive at a collective judgment.

In *When Does Approval Voting Make the “Right Choices”?*, Brams and Kilgour examine different contexts in which an individual voter approves of a proposal based on the proposal’s probability of being right (or good or just) and the voter’s probability of making a correct judgment of whether it is right (or wrong). For multiple proposals in which more than one proposal can be approved and under probabilistic settings, they determine conditions for when the most approved proposals have the greatest probability of being right, relating their results to the Condorcet Jury Theorem.

Nehring and Pivato take a more axiomatic approach to judgment aggregation in *How Indeterminate is Sequential Majority Voting? A Judgement Aggregation Perspective*. Similar to other choice problems, the order in which decisions are made can matter. This paper continues a research program exploring in exactly what ways the order in which judgments are made can affect the actual final judgments on various propositions, in this case showing that a very large number of natural examples exhibit various forms of this indeterminacy in sequential majority votes.

Several papers continue a long tradition in the field of using geometry and combinatorial objects to analyze a wide variety of voting questions. Stenson, in *Weighted Voting, Threshold Functions, and Zonotopes*, looks at a generalization of simple games and weighted voting systems. She explains how these games, and their winning/losing coalitions, correspond to vertices and other geometric aspects of particular polytopes and their duals. *The Borda Count, the Kemeny Rule, and the Permutahedron* by Crisman takes the geometric-combinatorial object called the permutahedron, and examines its symmetries with the goal of understanding social preference functions, which return not just winners but (sets of) rankings. The main result uses representation theory to characterize the most symmetric social preference functions as a one-dimensional continuum connecting the two rules in the title.

In *Double-Interval Societies*, Klawe, Nyman, Scott, and Su consider the effect of geometric constraints in a model in which voters approve of positions in a linear spectrum. When each voter’s approval set is represented by two disjoint closed intervals and when every pair of voters agree on some position, the authors determine a lower bound for the approval ratio. They also construct societies with low approval ratios by relating the double-interval models to the arrangement of n symbols in which each symbol appears twice, thereby relating the continuous geometry to discrete sequences.

By contrast, *Voting for Committees in Agreeable Societies* by Davis, Orrison, and Su focuses more on combinatorial aspects of agreeability, supposing that voters each approve of a certain size subset of candidates with a goal of selecting a committee of a possibly different size. Using a graph called the Johnson graph, they prove a number of results regarding what proportion of voters will be satisfied with the results under different suppositions about the distribution of voters’ intent.

Ratliff’s paper *Selecting Diverse Committees with Candidates from Multiple Categories* approaches committees from a different perspective. His focus is on selecting committees where candidates may be in several categories, and where it is desirable to have a committee with members from each of these categories. Using a combination of explicit examples and point-counting arguments with Ehrhart polynomials, both probabilistic and exact results regarding admissible ballot types to obtain these outcomes are derived.

Finally, we have three papers examining game theory from different mathematical perspectives. In *Expanding the Robinson-Goforth System for 2×2 Games*, Hopkins considers the relationships between bimatrix games in which outcomes are ordinally ranked, but there may be ties between outcomes. Robinson and Goforth developed a group-theoretic classification for the no-tie case; Hopkins revisits this work from a graph-theoretic perspective and shows that the edges of an associated graph yield information for when there is a single tie. He expands the Robinson and Goforth system using a collection of simplices that includes all 2×2 ordinal rank games with ties.

Although the decomposition of Jessie and Saari in *Cooperation in n -Player Repeated Games* may be viewed as a type of classification, it is more of a tool to analyze behavior in games. For two-strategy, n -player games, they decompose the games' n -matrix payoffs into behavioral and strategic components. For a specific game in a repeated setting, the strategic component captures the information necessary to determine Nash equilibrium behavior, while the behavioral component describes a type of cooperative behavior. This approach lends itself to the description of all games with any specific type of behavior, and how this behavior changes as the components change.

In a bankruptcy problem, a set of individuals have claims that when summed exceed the amount of an estate. A bankruptcy rule determines how to share the estate among the claimants. Motivated by the relationship between the two-player Contested Garment rule and the n -player Talmud rule, Jones and Wilson (*The Dynamics of Consistent Bankruptcy Rules*) define a dynamic averaging process in which a k -player rule is used on all subsets of size k , the outcomes are averaged, and the process is repeated. They show that when the k -player rule satisfies a well-studied notion of consistency, then the dynamic process converges to the n -player solution for any initial allocation.

In conclusion, as editors we want to thank all the authors for their interesting and strong papers. We enjoyed reading them, and hope that readers will experience that same pleasure in exploring the intersection between mathematics and the social sciences present in this volume.

Redistricting and district compactness

Carl Corcoran and Karen Saxe

ABSTRACT. The Supreme Court of the United States has held that bizarrely shaped congressional districts threaten the equal voting rights of U.S. citizens. Attempting to quantify this bizarreness, myriad measures of compactness have been produced. In recent years, a new class of measures has emerged that takes into account the geometric shape of the district, the geometric shape of the state, and also the spread of the population throughout that district. These measures are hybrids of path-based and population measures, and use census blocks as the basic unit for calculation. By hybridizing path-based and population measures, some deficiencies in measures that take into account only geometric shape or population dispersion are overcome. We conclude with a discussion of criteria for evaluating the quality of a compactness measure.

1. Introduction: Redistricting and Gerrymandering

According to the U.S. Constitution, every ten years a census must be taken to determine the population of the United States. From these data, seats in the House of Representatives are apportioned to the fifty states. This process is called reapportionment, and it determines how many legislators will represent each state in the House for the next decade. Once this number is determined, the states themselves are left to the task of determining which people are represented by each legislator.

Each legislator represents a single-member geographic congressional district. For states apportioned only one representative, the entire state is an at large district. But for states with multiple legislators, the state is carved up into districts. These districts are subject to conditions, however, and the most significant is equal population. In 1964, the Supreme Court decided in *Reynolds v. Sims* that all districts in a given state should have as nearly equal populations as possible. Since the population of each district undoubtedly changes from census to census, states must redraw these districts to ensure equal population. This process is known as *redistricting*.

2010 *Mathematics Subject Classification*. Primary 91-02; Secondary 91F10.

Much of this paper was developed for the undergraduate Honors Thesis of the first author.

Disclaimer: The second author served on the Minnesota Citizens Redistricting Commission (2011-2012); the views expressed here are her own, and do not necessarily reflect the views of the Commission.

The authors would like to thank Mathematics of Decisions, Elections, and Games AMS Special Session organizers Karl-Dieter Crisman, Michael A. Jones, and Michael Orrison, and all Session participants at the 2012 & 2013 Joint Mathematics Meetings.

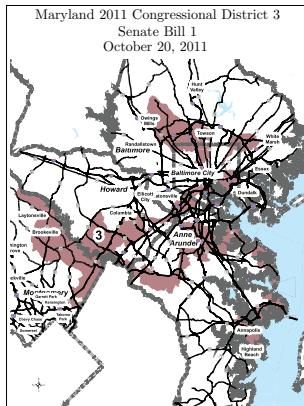


Figure 1. Maryland’s 3rd congressional district

As one might expect, redistricting can profoundly impact the political landscape of the state. With current methods and computational power, mapmakers can determine with a great degree of accuracy the political leaning of a given district, based on demographic data and election returns. This makes the redistricting process ripe for politically motivated manipulation. In most states, the power of district drawing is given to state legislators [11]. Each legislator naturally has an interest in how the districts are drawn, either to protect him or herself in state level elections, or to curry favor with his or her party. When this power is abused to give an advantage to one party or group, *gerrymandering* has occurred. Districts that re-elect incumbents who are protected by gerrymandering could well be districts that lean to partisan extremes (and therefore the individuals nominated could be left-leaning Democrats or right-leaning Republicans); this phenomenon could be resulting in our recent observed and increasing partisanship in the House of Representatives [3].

Scholars disagree on how precisely to define gerrymandering. Richard Morrill defines it as “the intentional manipulation of territory toward some desired electoral outcome,” [13], while to Michael McDonald and Richard Engstrom it is “the drawing of electoral districts so as to assign unequal voting weights to cognizable political groups.” [12]. These definitions underscore two critical facets of gerrymandering: the identification of communities of interest, and the (intended) consequences of the gerrymander. Taken together, they are a sufficient definition of gerrymandering.

Some scholars defend the gerrymander as an essential part of the redistricting machinery, but most identify its disenfranchising effects as an evil. Robert Stern identifies expressive harms caused by gerrymandering: it diminishes effective representation by decreasing the number of competitive districts, and it minimizes the need for coalition-building which would allow small, single-issue groups to be heard [21]. We agree with Stern, that gerrymanders do indeed threaten the effective representation that we strive for in the United States. How, then, are we to determine when a district has been gerrymandered? Furthermore, how do we prevent gerrymanders from being made in the first place?

The Supreme Court again has opined on the subject. In the 1982 case *Karcher v. Daggett*, it set forth criteria which were crystalized in 1993 with *Shaw v. Reno*.

These standards are known as “traditional districting principles,” and include, but are not limited to, equal population, compactness, contiguity, integrity of communities of interest, integrity of political subdivisions, and the integrity of natural boundaries. Extremely non-compact districts are generally taken to be gerrymanders. For example, on its face, Maryland’s 3rd district is definitely not compact; incidentally, it is also not contiguous (see Figure 1).¹

This paper examines the notion of compactness, which in essence is a measure of how spread out the parts of a district are. Many ingredients can go into a compactness measure, including perimeter, population dispersion, and area dispersion. Section 2 includes a very brief survey of traditional compactness measures together with an exposition of a newer class of measures. Section 3 assembles criteria for a quality compactness measure, offers a discussion of the quality of various measures according to these criteria, and reflects on the utility of compactness measures in practice.

2. Measuring Compactness

2.1. Traditional Compactness Measures. In the past few decades, the use of computers in redistricting has exploded, to the point where every state now uses some form of software to make their maps [12]. Built into many of these software packages are several measures of compactness. This accessibility, combined with ever-increasing computing power, has made compactness a routine evaluation when creating a districting plan. However, there remains the question of which measures to use, and which measures are the ‘best.’ There is and can be no consensus on this point, as every measure has both strengths and limitations.

In previous surveys of compactness measures (for example, see Young 1988, Niemi et al. 1991), the authors come to this same conclusion. Instead of prescribing a specific measure to use, they explore the benefits and detriments of several. Niemi et al. classify these measures into three types: perimeter measures, area dispersion measures, and population dispersion measures. We now present a simple example of a measure from each of these three categories. The first measure is a perimeter measure, the second is an area dispersion measure, and the third is a population dispersion measure. We choose the three as they are conceptually simple, and were all three used in our state of Minnesota during the 2011-12 round of redistricting.

2.1.1. *A Perimeter Measure: Polsby-Popper.* Polsby and Popper [16] introduced their measure of compactness as a variation on Schwartzberg’s measure [20]. They write, “[t]he absolute measure of a shape’s efficiency is determined by dividing the area of the shape by the area of a circle with perimeter of equal length.” The Polsby-Popper score is thus given by

$$PP(D) = \frac{4\pi A(D)}{p^2},$$

where D is a district, p is the length of the perimeter of the district, and $A(D)$ is the area of the district. Though it is not how it was originally defined, it turns out that Schwartzberg’s score is the square root of the reciprocal of this.² We choose to focus on Polsby-Popper since its scores are always in the interval $[0, 1]$.

¹http://www.planning.maryland.gov/PDF/Redistricting/2010maps/Cong/Dist_3

²It is defined as the ratio of the perimeter of a district to the perimeter of a circle with equal area.

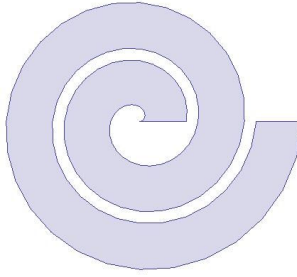


Figure 2. Hypothetical district meandering within a circular shape

2.1.2. *An Area Dispersion Measure: Reock.* The Reock test [17] amounts to finding the smallest circle containing the district and taking the ratio of the district’s area to the area of the circle. We let $D \subset \mathbb{R}^2$ be a district, and again let $A(D)$ be the area of the district. Then, let S_C be the set of all circles in \mathbb{R}^2 that completely contain the district D . Define the circle C_{min} to be such that $A(C_{min}) = \inf\{A(C) | C \in S_C\}$, where $A(C)$ is taken to be the area of the region enclosed by the circle C . The value of the Reock score of a district D is then given by

$$\text{Reock}(D) = \frac{A(D)}{A(C_{min})}.$$

If the district is a union of islands, smallest circles are found for each island.[2]

Both the Polsby-Popper and Reock measures have values between 0 and 1, with scores closer to 1 indicating more compact districts. This makes the scores readily comparable across plans. However, they both presume the circle to be the most compact shape. While this makes sense geometrically, it doesn’t so much in the context of redistricting. No state can possibly be covered with a finite number of non-overlapping circular districts. Moreover, districts that meander within a vaguely circular shape will score highly with Reock, even if they are facially quite non-compact (see Figure 2).

2.1.3. *A Population Dispersion Measure: Population Circle/Convex Hull.* The population circle measure is the ratio of the district population to the population contained in the smallest circle containing the district (defined the same way as with the Reock test). The convex hull measure simply replaces the smallest circle encompassing the district with the convex hull of the district. The convex hull measure is referred to as the “rubber band” measure and was first discussed in [9].³

These two measures are much better suited to detecting urban gerrymanders, versus rural. For instance, in a region with low population density, except for small pockets of dense habitation, a gerrymander that connects these pockets will score highly by this measure, as few people live in the areas cut out by the gerrymander. Moreover, the meandering circle district of Figure 2 will score a perfect 1 if the population is completely contained in the shaded area, and will even score highly if the population is merely concentrated in the shaded area.

³Existing *Mathematica* code is written for several computations of this paper. For example, S.J. Chandler’s *A Minimal Circumcircle Measure of District Compactness* <http://demonstrations.wolfram.com/AMinimalCircumcircleMeasureOfDistrictCompactness/> can be used to find the convex hull measure of a given district.

2.2. Path-based Population Hybrid Models. In recent years, a new trend in measuring compactness has emerged. Path-based measures look at the path connections of a given district. Here, we define a path in a mathematical sense: a path γ is a continuous mapping of the real interval $[0, 1]$ to \mathbb{R}^2 . For the purposes of evaluating compactness of congressional districts, two types of paths stand out: road-based paths, and shortest Euclidean distance paths. This idea, of examining different types of paths, is also explored in the recent work of Chambers and Miller [6].

Roads have two pragmatic benefits as a foundation for a path. First, roads are the most practical means for a congressperson to physically reach their constituents. Second, roads tend to avoid aberrant natural phenomena, such as steep mountains or large lakes; the shortest Euclidean path pays no attention to these. But the role that roads play in districting should be considered somewhat anachronistic. Political communication is less dependent on face-to-face discussion than ever before in our history, and representative government has become less and less dependent on physical presence. Furthermore, roads do not necessarily reflect voter proximity – an undoubted goal of population measures – as they can meander without respect to voter inhabitation. For these reasons, the shortest Euclidean path is the preferred choice for our path-based measure.

One notable measure that is unclassified (though mentioned) by Niemi et al. is that of Papayanopoulos. His measure, developed in 1973, sums all pairwise distances of census enumeration district centers weighted by their populations [15]. After 1973, census enumeration districts were phased out and have been replaced by census blocks. By the 1990 census, the area of the United States was covered entirely by census blocks, and no longer census enumeration districts [4]. Let D be a given district made up of k census blocks b_1, b_2, \dots, b_k . Let d_{ij} be the length of the line segment between the geographic centers of blocks b_i and b_j . As well, let p_i denote the population of census block b_i . Then, the value of the measure is given by

$$\text{Papa}(D) = \sum_{b_i \in D} \sum_{b_j \in D} d_{ij} p_i.$$

We view Papayanopoulos’s measure as a precursor to a wholly different class: path-based measures.

The next example of a path-based measure comes from Fryer and Holden. They create a measure called the “relative proximity index,” which measures “distance between voters within the same political district in a state relative to the minimum such distance achievable by any districting plan in that state.” [7] Given a state S , a finite collection of subsets of S , $S_D = \{D_1, D_2, \dots, D_k\}$, is called a *districting plan* of S if S is the disjoint union of the sets in S_D ; that is $S = D_1 \dot{\cup} D_2 \dot{\cup} \dots \dot{\cup} D_k$. Consider a voter v (or u) to be a constituent element of D_i and define

$$\pi(S_D) = \sum_{D_i \in S_D} \sum_{v \in D_i} \sum_{u \in D_i} (d_{vu})^2,$$

where d_{vu} is the Euclidean distance function between the locations of voters v and u . Next, let S_D^* be the districting plan that minimizes π . The relative proximity index is then defined as

$$RPI(S_D) = \frac{\pi(S_D)}{\pi(S_D^*)}.$$

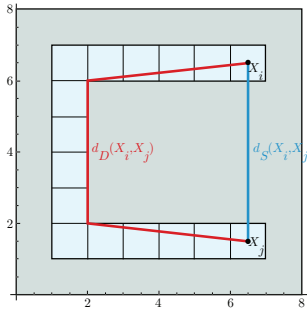


Figure 3. The distance ratio $d_S(x_i, x_j)/d_D(x_i, x_j)$

Note that the value of this measure ranges from 1 to infinity, with higher scores indicating more severe non-compactness. Fryer and Holden also provide a method, based on Voronoi diagrams, for finding S_D^* .

The contiguity requirement in redistricting ensures that every district will be path connected. However, when the nature of a given path is considered, there is ample room to make judgements about the irregularity of a district. Papayanopoulos and Fryer and Holden use the straight line distance between, respectively, geographic centers and individual voters, even if this straight line path does not lie entirely within the district under consideration. This observation becomes a priority in our further discussion of path-based measures.

To establish notation, consider a geographic legislative district D in a state S , made up of k census blocks and with a total population of P . We take b_i to be the i^{th} census block, with population p_i and geometric center x_i , for each integer i between 1 and k . The distance between x_i and x_j within the district D is denoted by $d_D(x_i, x_j)$.

Recall that Maryland’s panhandle involves a jagged coastline. Any district in that panhandle must also be bounded by the coastline. This may exaggerate the distance $d_D(x_i, x_j)$, when in reality the mapmaker is given no better option.⁴ To account for this lack of choice, we also consider $d_S(x_i, x_j)$, the distance between x_i and x_j within the state. For example, consider a “C” shaped district composed of square census blocks. Figure 3 gives an example of the distances that d_D and d_S describe, indicating that the former might be significantly greater than the latter. Here, the light blue (lighter gray if reading the black and white version of this paper) region represents the district D , and the gray (darker gray) region represents the state S .

We are then concerned with the ratio of these two distances:

$$\frac{d_S(x_i, x_j)}{d_D(x_i, x_j)}.$$

Note that the value of this ratio is bounded above by 1. We use this ratio as a weight for the sum of the i^{th} and j^{th} populations. Then, we sum these pairwise

⁴District boundaries are regularly drawn as straight lines through bodies of water specifically to minimize total perimeter. For an example, see http://planning.maryland.gov/PDF/OurProducts/Redistrict/CongDist/District/Color_Map/CongDistColor100Dist.shtml

ratios for each b_i and b_j :

$$\sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} (p_i + p_j).$$

A question becomes apparent at this point: What is to be done when b_i and b_j are the same census block? When this occurs, the distance ratio is not defined. We define it to be 1, and explain this choice after discussion of Chambers and Miller's class of measures.

Looking again to $\sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} (p_i + p_j)$, the maximum value this sum can take is $2kP$, where k is the number of census blocks in D . That is to say, if we have a convex figure, $\frac{d_D(x_i, x_j)}{d_S(x_i, x_j)} = 1$ for all i and j . It follows that

$$\sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} (p_i + p_j) \leq \sum_{b_i \in D} \sum_{b_j \in D} (p_i + p_j) = \sum_{b_i \in D} (kp_i + P) = kP + kP = 2kP.$$

Therefore, we can force the value of the expression between 0 and 1 by multiplying by $(2kP)^{-1}$. This gives us our measure of compactness, which we will call C :

$$C(D) = \frac{1}{2kP} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} (p_i + p_j),$$

where, to recall, D is a geographic legislative district in a state S , made up of k census blocks and with a total population of P , and b_i is the i^{th} census block, with population p_i and geometric center x_i , for each integer i between 1 and k .

With this measure, scores all lie between 0 and 1, scores closer to 1 indicate a more compact district, and all convex figures score a perfect 1.

To our knowledge, the first exposition of path-based measures as a unique class comes from the recent work of Chambers and Miller. They describe a family of measures that centers on the expected relative difficulty of traveling between two points in the district [5]. Aside from their general family of measures, they produce one in particular. They define a piecewise function $d(x, y)$ that has value 1 if the shortest path between two given points x and y is contained in the district, and 0 otherwise. Their measure then, is given by the double integral

$$\int_D \int_D d(x, y) \frac{f(y)f(x)}{(F(D))^2} dy dx,$$

where D is the district, f is the true population density function, and $F(D) = \int_D f(x) dx$. More generally, they give a parametrized family of measures:

$$\int_D \int_D \left[\frac{d_S(x, y)}{d_D(x, y)} \right]^q \frac{f(y)f(x)}{(F(D))^2} dy dx,$$

where the parameter q is to take on any non-negative value. However, census data is often not detailed enough to yield such a density function. As an approximation, they give a discrete version of their measure

$$s^q(D) = \left[\sum_{b_i \in D} \sum_{b_j \in D} p_i p_j \right]^{-1} \left[\sum_{b_i \in D} \sum_{b_j \in D} \left[\frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} \right]^q p_i p_j \right].$$

The measure that we consider is apparently very similar to Chambers and Miller’s discrete version for $q = 1$, with $p_i + p_j$ replacing $p_i p_j$ in each term. Consistent with the first integral form, they proceed to only consider the measure $s^q(D)$ for $q = \infty$, taking

$$\left[\frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} \right]^\infty = \begin{cases} 1 & \text{if } \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Chambers and Miller do not address the question of what is to be done when b_i and b_j are the same census block and the ratio becomes indeterminate. But, we can interpret what the necessary definition is. One obvious candidate is to define the ratio to be zero. However, this would imply the Chambers and Miller measure varies between 0 and some rational constant less than 1. Suppose we define $\frac{d_S(x_i, x_i)}{d_D(x_i, x_i)}$ to be 0. We observe that for a convex district D , their discrete measure $s^1(D)$ is, then,

$$\left[\sum_{b_i \in D} \sum_{b_j \in D} p_i p_j \right]^{-1} \left[\sum_{b_i \in D} \sum_{b_j \in D} p_i p_j - \sum_{b_i \in D} p_i^2 \right] = 1 - \frac{1}{P^2} \sum_{b_i \in D} p_i^2.$$

Furthermore, this value changes given the populations of each b_i , as well as the number of census blocks in the district. Thus, the upper bound on the measure would vary for different districts. This would certainly be an undesirable feature of a compactness metric, so we can conclude that defining the distance ratio to be 0 when $i = j$ is unreasonable. Instead, it is preferable to define it to be 1, as this would place the value of Chambers and Miller’s discrete measure in the interval $[0, 1]$, and take the value 1 for a convex D . We choose to do the same for our measure.

While Chambers and Miller define the class of measures for parameter $q \geq 0$, they only give further discussion of the case $q = \infty$. To emphasize, the special case $q = \infty$ includes a non-zero contribution for x_i and x_j in the sum if and only if the shortest path between the two points actually lies in the district. Also note that if either block b_i or b_j has no one living in it, then the corresponding term gives no contribution since it is the product of the populations that is used (this is independent of the value of $q \geq 0$). One may think this is irrelevant but, in fact, a significant proportion of census blocks have population zero. In Minnesota, the authors’ home state, the 2010 census recorded 107,794 of the 259,778 blocks as having zero population.⁵ Despite the similarities between our measure and Chambers and Miller’s $q = 1$ measure, the scores diverge for at least one important type of example, as will be described in the next section.

Finally, Hodge, et al. take the notion of path-based compactness in a different direction. They develop a convexity coefficient measure that instead of looking at pairwise distances, looks at the “visible” area from a given point [8]. They define the “visible region” $V_D(x, y)$ of a point (x, y) in a district D to be the set of all points (x', y') in D such that the shortest straight line path between (x, y) and (x', y') is wholly contained in D . Then, the fraction of the district’s area that this visible region encompasses is computed for each (x, y) , and summed, yielding the

⁵<http://www.gis.leg.mn/html/download.html>

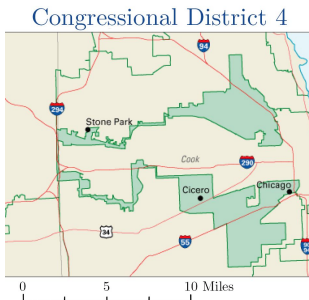


Figure 4. Illinois’s 4th congressional district

double integral

$$\chi(D) = \int_D \int_D \frac{A(V_D(x, y))}{(A(D))^2} dx dy,$$

where $A(V_D(x, y))$ is the area of the visible region of (x, y) , and $A(D)$ is the area of the district. For each of the 435 U.S. congressional districts, they estimate the value of this convexity coefficient using Monte Carlo methods to approximate the double integral.

As thus defined, this measure does not factor in the population dispersion of a given district, nor does it take into account irregular state borders. They modify this measure in two ways, first to account for irregular state borders, and then to account for population distribution within the district. In order to account for the population distribution, they replace random points with points selected to represent random census blocks. We are not confident about the full implications of this choice, but it can be said that in Minnesota (and many other states), a random selection of census blocks would result in a large number of blocks chosen with no inhabitants. In Minnesota, between one-third and one-half of the blocks chosen using their method would have no inhabitants.

2.3. Examples. The aim of developing path-based population hybrid measures is to take into account both the state boundary and also distribution of population within the district. Hence, we now look at how the measures are affected by changes in either the population of the census blocks, or the border of the state. We use hypothetical “C”-shaped districts to illustrate these changes (see Figures 5 and 6). While this “C” shape is a model, Figure 4 shows that there are in fact districts that resemble our hypothetical district.⁶

In each model “C”-shaped district, the light blue (lightest gray if reading the black and white version of this paper) area is the “district,” and the gray (medium gray) area is the “state.” In each example, the district is broken into square census blocks, and the population labels each block. For convenience, every district is taken to have total population 1600. In examples 2 and 3, the dark blue (darkest gray) area is water, creating a boundary for the “state.” All three examples of Figure 5 show a uniformly distributed population in this “C” shape. Examples 2 and 3 show how the boundary of the state near the boundary of the district affects the compactness score. For each example, we calculate the compactness score for several of the compactness measures described. “Corcoran” denotes the measure

⁶http://nationalatlas.gov/printable/images/pdf/congdist/IL04_110

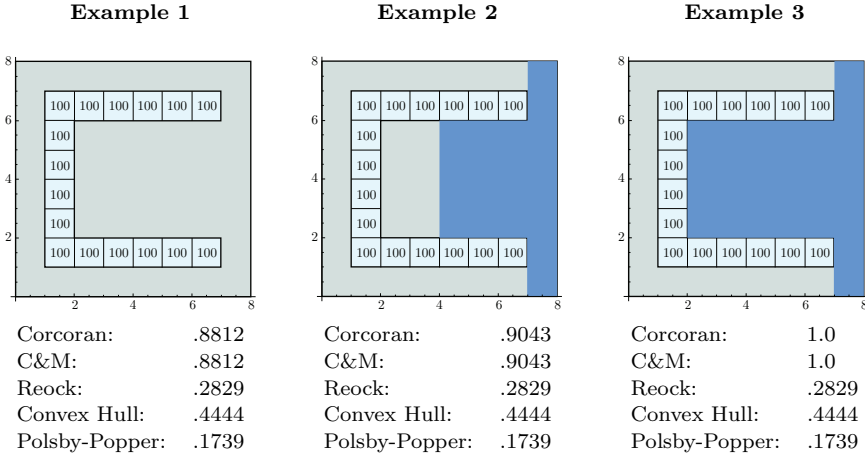


Figure 5. Comparing measures, uniform population dispersion

introduced in the previous section, and “C&M” denotes the discrete version of Chambers and Miller’s measure evaluated for $q = 1$, and will be referred as simply the “Chambers and Miller measure” for the remainder of the section.

Example 3 demonstrates that when a district follows a state border, when given no other choice, both path-based measures (Corcoran, Chambers and Miller) score perfectly when there are no other irregularities. The other three measures do not take into consideration the state border.

When the population is evenly distributed, both Chambers and Miller’s measure and our measure will score any given district identically. Suppose that in a district D , every census block b_i has the same population p . Then, $P = kp$ and hence this paper’s measure will give the score

$$\frac{1}{2kP} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} (p + p) = \frac{2p}{2k^2p} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} = \frac{1}{k^2} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)}.$$

Similarly, the Chambers and Miller measure yields

$$\frac{1}{P^2} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} p^2 = \frac{p^2}{k^2 p^2} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)} = \frac{1}{k^2} \sum_{b_i \in D} \sum_{b_j \in D} \frac{d_S(x_i, x_j)}{d_D(x_i, x_j)}.$$

So, if each census block has the same population, there will be no difference in the measures’ scores. However, while it is an interesting diversion, such a case is hardly of practical concern. It is difficult to imagine a congressional district where population is evenly distributed across census blocks.

If we turn our attention to the same “C” shaped district, but instead consider non-uniform population distribution, we begin to see differences between our measure and Chambers and Miller’s measure (Figure 6). Examples 4 and 5 should be compared to Example 1, above. Again, the other three measures fail to capture any change.

In Example 5, we see that Chambers and Miller’s measure fails to penalize for population dispersion. Indeed, as long as the population is concentrated in a convex portion of the (most likely non-convex) district, Chambers and Miller gives a perfect score of 1, while our measure identifies a potential manipulation of boundaries. Some would argue that an irregular district boundary doesn’t matter

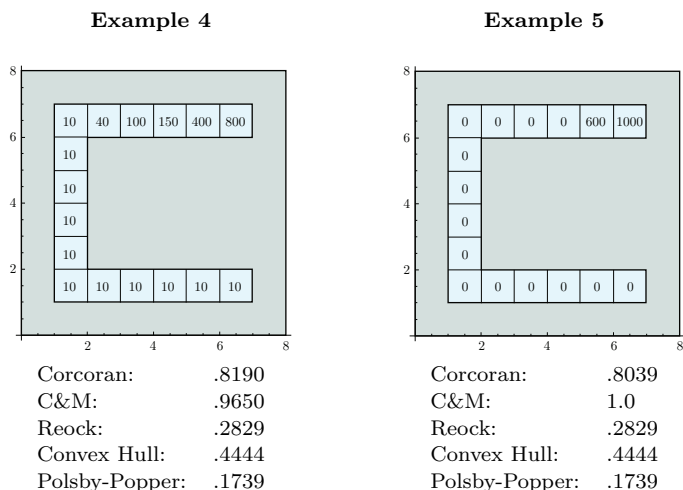


Figure 6. Comparing measures, non-uniform population dispersion

in this case, but we believe that it can be cause for concern. For example, district boundaries usually stay in place for a decade; it is possible to predict development and growth (perhaps most notably in suburbs of large metropolitan areas) over such a time frame and those in charge of redistricting may be privy to information that could give motive for such line drawing. Second, in many states, line drawers are bound to keep new districting plans as close as possible to old ones. In other words, a strange empty, reaching arm drawn today could become a strange populated arm in the future (and vice versa).⁷ Last, it is a challenge to find a districting plan with a unique gerrymandered district; if one district is bizarrely shaped, chances are that other nearby districts are too.

We end this section by remarking that a fair criticism is that the path-based measures in Figures 5 and 6 give values that are very high (close to 1). Taking different values of the parameter q (which can be done for both Corcoran and Chambers and Miller) will result in a wider spread of values, and taking $q = \infty$ does so most dramatically.

3. Criteria for Compactness Measures and Discussion

In the redistricting process, there are three of the traditional principles, as described in our Introduction, that must be upheld by each state: population equality, contiguity, and compactness. Given a districting plan, it is straightforward to judge population equality and contiguity. There are many ways to measure compactness, and different measures have relative weaknesses and strengths. For example, the Polsby-Popper measure discussed in Section 2.1.1 treats only a circular district as a ‘perfect’ district, even a square district would be considered less than ideal using this measure alone. The recognition that using one measure alone is unwise is evidenced by the fact that redistricting groups often require analysis of proposed plans using several measures. With path-based measures, while we have gotten away from the notion that a circle is the perfect compact district shape, we do have the problem

⁷See, for example, Section IV (page 9) of *Final Order Adopting A Congressional Redistricting Plan*, Feb 21, 2012, found at <http://www.mncourts.gov/?page=4469>



Figure 7. California with rectangular district

that any convex district is ranked highly. For example, an elongated rectangle in California, stretching the entire state from southeast to northwest, would be given a high score with the path-based measures discussed herein (Figure 7). How should we evaluate the quality of measure(s) of compactness? Which measures should be adopted during the redistricting process?

There have been several surveys of compactness measures, notably Young (1988) and Niemi, et al. (1991) (see also [1], [7]). Young surveyed eight different tests of compactness, ranging from the simple “visual test” to the more sophisticated Reock test. After looking at each of these measures and determining their strengths and weaknesses, Young concludes with five desirable properties of compactness measures (Table 1).

TABLE 1. Young’s Criteria [22]

1.	There is no single threshold value which determines when a district is no longer compact. Instead, measures should be used to compare plans.
2.	Measures should apply to districting plans as a whole, and not just to individual districts.
3.	A measure should treat political subdivisions and census tracts (now ‘census blocks’) as indivisible units whose shape does not affect the measure.
4.	A test should measure the shape of the district, not the size.
5.	A measure should be conceptually simple and require only easily collectible data.

While far from exhaustive, each of these criteria describes a desirable facet of measuring compactness. The first criterion suggests that compactness scores are merely indicators of gerrymandering. As Figure 8 illustrates, gerrymandering can occur with shapes that remain, at least facially, fairly compact. It is not the value of a particular score that gives information about gerrymandering. Instead, we are interested in the relative ordering of different plans determined by that particular score.

The second criterion is a much more pragmatic one. In redistricting, compactness is most often applied when developing a plan, or when adjudicating a gerrymandered district. Due to the reluctance of federal and state courts to inject themselves in the political process of redistricting and the perceived difficulty in fashioning an appropriate judicial remedy, gerrymanders other than the most

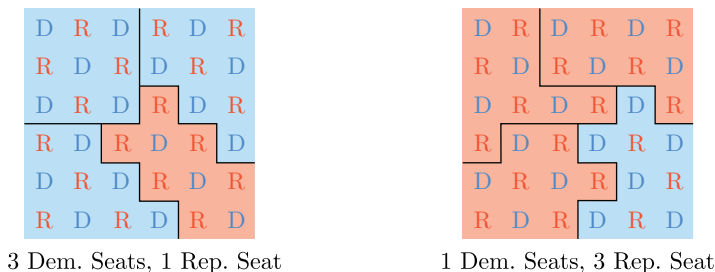


Figure 8. Gerrymandering example

egregious are rarely the subject of litigation. Thus, it could be argued that the primary use of compactness scores is in the creation of a districting plan which must, necessarily, be considered as a whole.

The third criterion stems from another principle of redistricting, that of preserving integrity of communities of interest and of political subdivisions. Under this criterion, population-based measures are preferable to measures based only on geometry.

The fourth criterion prevents discrimination between large rural and small urban districts, as rural districts are necessarily large to ensure equal population. This criterion alone suggests that the so-called Perimeter measure, whose value is merely the perimeter of the district, is a poor choice.

The fifth criterion, while still desirable, is anachronistic insofar as computers have become more and more powerful, and redistricting software more and more common. However, overly complicated measures can be difficult to implement and interpret, which means that only a small cadre of experts can be relied upon to detect and evaluate gerrymanders. As a result, this criterion helps ensure that transparency remains a priority in the redistricting process.

Niemi et al.’s survey two years later had broader scope, examining a longer list of existing measures. Furthermore, they classify the measures based on the main ingredient of compactness they utilize. After categorizing the measures, the authors, like Young, reflect and give criteria for compactness measures (Table 2).

TABLE 2. Niemi et al.’s Criteria [14]

1.	Comparisons of compactness scores should not be made across states.
2.	There should be no threshold value, but measures are instead to be used for comparisons.
3.	Compactness should not be the only criterion applied when judging a districting plan.
4.	Multiple measures should be used whenever possible.

For the first criterion, one need only look at Colorado and Maryland side by side to justify its inclusion. For nearly every measure, the districts of Maryland will be less compact than the districts of Colorado. Maryland, of course, has a jagged, incising coastline which skews the score of most compactness measures. But these are forgone conclusions, as state borders do not change and congressional districts are subject to these boundaries. So, using most traditional compactness measures, comparisons across states are inappropriate. With the introduction of

path-based measures, we can examine anew the importance of this criterion and start to compare scores across states.

The second criterion is essentially the same as Young’s first criterion.

The third criterion is again a standard application of traditional districting principles, which have been affirmed and reaffirmed by the Supreme Court.

The fourth criterion is a commonsense acknowledgement of the imperfection of compactness measures applied alone. As the meandering district of Figure 2 and the rectangular district in California of Figure 7 demonstrate, there are classes of shapes that each compactness measure fails to identify as non-compact. So, scoring poorly by one measure should not be cause for concern. However, a district that scores poorly by several compactness measures is clearly less compact than one that scores poorly by only one measure. Thus, the use of multiple measures is a practical safeguard against improperly identifying a district as compact or non-compact.

Broadly speaking, these criteria can be classified into two categories: implementation criteria and design criteria. The former are used when creating and/or evaluating a districting plan; the latter are used when evaluating the quality of a measure itself. The implementation criteria are Young’s first and second, and all four of Niemi et al.’s criteria. The design criteria are Young’s third, fourth, and fifth. Even though the traditional measures can be applied successfully according to the implementation criteria, many do not stand up well with respect to the design criteria. The emergence of the class of path-based population hybrid measures gives evidence that the design criteria are gaining importance as this field develops and matures.

Assessing compactness is further complicated by a lack of acknowledgement of implementation criteria by state and local entities. In practice, most states require districts to be compact, but many give no particular measure or measures to be used.

We now turn to the recent bout of redistricting in Minnesota for an illustration of what can happen. On November 4, 2011 an order⁸ “stating principles and requirements for Minnesota plan submissions” was filed, and requires inclusion of a report

stating the results of Reock, Schwartzberg, Perimeter, Polsby-Popper, Length-Width, Population Polygon, Population Circle, and Ehrenberg measures of compactness for each district.

Consider two plans submitted to the Minnesota Commission, dubbed *Hippert* and *Martin* (Table 3).

Recall that, for each measure, numbers closer to 1 indicate a more compact district. Note that the plans look equally good with Reock, Polsby-Popper favors Martin, while the population circle (and necessarily Schwartzberg) favor Hippert. In other words, we can have two plans A and B for a given state, and two compactness measures m_1 and m_2 , so that A is more compact than B using m_1 , yet B is more

⁸In spring of 2011, the MN legislature presented a state redistricting plan to Gov. Mark Dayton who quickly vetoed it. A gubernatorial veto also occurred in 3 of the last 4 decades (1971, 1981 and 2001). In response to the veto, the MN Supreme Court Chief Justice appointed a special 5 member Judicial Commission to come up with a plan by February 21, 2012, in case the legislature continued to fail to produce a plan that the Governor could approve. The November 4 order was given by the Judicial Commission 3 months prior to their deadline for choosing a plan, and only 2 weeks before their November 18, 2011 deadline for receiving plans from groups. The order can be found at <http://www.mncourts.gov/?page=4469>

TABLE 3. *Hippert* and *Martin* plans for Minnesota congressional districts

	Reock	P-P	Pop. Circle	Schwartzberg
Hippert	0.37	0.30	0.33	1.70
Martin	0.37	0.31	0.30	1.74

Minnesota’s eight congressional districts each receive a score by a given measure, and the number in the cell is the mean of eight scores for the measure.

compact than A using m_2 . This example illustrates the reason why Niemi et al.’s fourth criterion is important, but also shows that the use of multiple measures might in fact lead to confusion within the political conversation.

On the one hand, Minnesota meets the implementation standards of Young and Niemi et al. However, the measures used could be criticized for their design. For example, the Perimeter score for a given district is simply the perimeter of that district and clearly violates Young’s fourth criterion.

The Supreme Court’s 1993 endorsement of traditional districting principles in *Shaw v. Reno* solidifies the role that compactness measures will play in future redistricting. Most existing measures are geometrically based, and some take into consideration the spread of the population within the districts. We agree with previous authors that compactness measures should be used only to compare amongst plans, and several measures should be applied; they cannot be used to determine whether or not a single district has been gerrymandered.

In terms of the criteria as outlined by Young and Niemi et al., how do path-based population hybrid measures hold up? These measures consider the shape – and not size – of districts; they can be used to measure an entire plan (by considering the range and distribution of scores for districts in the plan); census blocks are used and block populations and geographic centers are freely available online. Importantly, these measures stand apart from the other measures in that they can be used to compare across states, since state borders are taken into account.

In the end, compactness cannot ensure fair representation. This said, measures of compactness should be used to assess districting plans, in conjunction with other tools. Further, it may be that, in the future, information about the demographics of the district population (besides simple distribution) will play increasingly important roles as ingredients in compactness measures. A strength of all path-based measures, that iterate over voters or census blocks, is the potential for population data stratification. The idea is that instead of trying to find gerrymanders in general, we can look for specific types of gerrymanders. Instead of simply using raw population, we can stratify by political leaning, race, or any other cognizable political grouping that can be used in gerrymandering. For instance, a score can be computed for only Democrats in the district. This can then be compared to a score for Republicans, or raw population. If the Democratic score is significantly lower, it may indicate a Democratic gerrymander; the shape of the district may have been manipulated to pack more Democrats into the district at the expense of compactness. While this causal relationship might not always hold, the potential for stratification could have a great impact on gerrymander identification. Thus, there may be more states that move away from using simply geometrically-based compactness measures as a way to control and detect gerrymandering.

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MATHEMATICS DEPARTMENT, ST. PAUL ACADEMY AND SUMMIT SCHOOL, ST. PAUL, MINNESOTA 55105

E-mail address: `ccorcoran@spa.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, MACALESTER COLLEGE, ST. PAUL, MINNESOTA 55105

E-mail address: `saxe@macalester.edu`

Fair division and redistricting

Zeph Landau and Francis Edward Su

ABSTRACT. Recently, Landau, Reid, and Yershov provided a novel solution to the problem of redistricting. Instead of trying to ensure fairness by restricting the shape of the possible maps or by assigning the power to draw the map to nonbiased entities, the solution ensures fairness by balancing competing interests against each other. This kind of solution is an example of what are known as “fair division” solutions—such solutions involve the preferences of all parties and are accompanied by rigorous guarantees of a specified well-defined notion of fairness. In this expository article, we give an introduction to the ideas of fair division in the context of this redistricting solution. Through examples and discussion we clarify how fair division methods can play an important role in a realistic redistricting solution by introducing an interactive step that incorporates a certain kind of fairness that can be used in concert with, and not as a substitute for, other necessary or desired criteria for a good redistricting solution.

1. Introduction

Redistricting is the political practice of dividing states into electoral districts of equal population. It is mandated to occur every ten years, after the census, to ensure equal representation in the legislative body. Where the boundaries are drawn can dramatically alter the number of districts a given political party can win. As a result, a political party which has control over the legislature, can (and does) manipulate the boundaries to win a larger number of districts, thus affecting the balance of power in the U.S. House of Representatives. This kind of boundary manipulation occurs even with certain legal and legislative constraints that restrict some aspect of how districts can be drawn and mandate that, where appropriate, districts should be created to have a majority of voters consisting of a racial minority. (See [11] for a detailed summary of these constraints.)

The ability of one political party to gain political advantage by carefully choosing the boundaries during redistricting has been recognized as a serious problem with the redistricting process in the United States; we shall refer to this as the problem of *partisan unfairness*. Attempts to fix and/or mitigate the problem of partisan unfairness (beyond the legal restrictions) have taken one of two approaches: trying to constrain the process to reduce the amount of political gain achievable, and

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trying to remove politics from the redistricting process. Examples of the first approach include attempting to limit the power of the drawing party by more strictly prescribing the allowable shapes of districts, and banning the use of registration and voting data within the redistricting process. Examples of the second approach include assigning the task of redistricting to bipartisan or non-partisan panels¹, and using computer programs to generate redistricting maps that optimize certain carefully chosen criteria.

Landau-Reid-Yershov [6] took a different approach to provide a novel solution to the problem of partisan unfairness: rather than trying to fix the problem by restricting the shape of the possible maps or by assigning the power to draw the map to nonbiased entities, their solution ensures fairness by balancing competing interests against each other. This kind of solution is an example of what are known as “fair division” solutions—such solutions account explicitly for the preferences of all parties, are determined by a procedure in which all parties are actively involved, and are accompanied by rigorous guarantees of a specified notion of fairness.

The goal of this article is to provide an exposition of this redistricting method in the context of a detailed sample “map”, and make a stronger connection to the ideas of fair division than is provided in [6]. In particular, we propose a specific notion of fairness that was used but not made explicit in [6]. This notion of fairness can be used in concert with (not substitute for) other necessary or desired criteria for a good redistricting solution. We clarify how fair division ideas can play an important role in a realistic redistricting solution by introducing an interactive step that involves multilateral evaluation, procedural fairness, and fairness guarantees. And by making the bridge between fair division ideas and redistricting solutions more explicit, we hope to encourage the flow of ideas between the two areas.

We begin with an introduction to the ideas of fair division in Section 2. We describe the problem of partisan unfairness in more detail in Section 3. Despite the ability to easily recognize the “unfairness” inherent in the redistricting process, it has been hard to give a reasonable definition of what would be fair. In Section 4 we give an explicit definition of fairness—the *geometric target*—that incorporates geometric considerations such as how constituent voters are distributed and how districts are shaped. Section 6 examines the protocol of [6] by analyzing its behavior in a specific example that demonstrates most aspects of the solution. Within this example, the fairness of the protocol is discussed in detail.

We stress that the ideas discussed here are well suited to be combined with other necessary or desired criteria for a good redistricting solution. Our definition of fairness involving geometric targets in Section 4 can incorporate independently desired requirements for district shape or competitiveness. The protocol of [6] can be easily adjusted to approximate fairness under these additional requirements. If a solution involving an independent commission is desired, these definitions of fairness can be used as a target for the commission. Similarly if a computer assisted solution that consists of optimizing some function is sought, this measure of fairness can be used as a component of the function to be optimized. Separately, the simple fair division ranking protocol given in Section 5 can be used to incorporate some degree of legislative preference within any proposed redistricting solution that otherwise does not include legislative influence.

¹Eleven states redistrict using bipartisan or independent commissions.

2. Fair Division

The problem of *fair division*, as Steinhaus [14] put it, is essentially a question of how to divide some object *fairly*. Usually, this object is affectionately referred to as *cake* [12], but in general it could be desirable or undesirable (e.g., the division of *chores* [9]) or a mixture of desirable and undesirable goods [15]. The cake may be infinitely divisible (as we usually regard real cake) or only divisible into discrete pieces (such as a pieces of an estate). Applications of fair division ideas include methods for resolving international disputes and divorce settlements [3].

There are several notions of *fairness* that one might consider, but an important aspect of fair division problems is that this fairness notion is evaluated by the parties involved in the negotiation, rather than an outside arbiter. Thus, the outcome of a fair division procedure will give each party what it considers to be a fair share, according to its own evaluation.

The simplest example of a fair division procedure is the familiar “I-cut-you-choose” method for dividing a cake among two people. One might consider a fair piece in which each party does not envy the other; we call such a solution an *envy-free* solution. Again, note that envy is measured by each party according to its own evaluation. If one person cuts the cake (into two pieces that she is indifferent between) and the other person is allowed to choose first (picking the piece that he most desires), then both people will end up with a piece for which they experience no envy. This simplest of all fair division procedures already highlights some interesting features common to all fair division procedures:

- (1) *Multilateral evaluation.* Fairness is evaluated according to each party’s own preferences. Therefore, parties don’t have to agree on what is valuable; each will obtain a share they would consider fair in their own estimation (and they do not need to know the other party’s preferences).
- (2) *Procedural fairness.* There is a process by which preferences are elicited, all parties are involved in the process, and they understand the criteria by which fairness is measured. Because of this, parties are more likely to feel that the process is fair, more so than a decision imposed by an outside arbiter (see e.g., [13]). The procedure guides parties to a mutually acceptable division.
- (3) *Fairness guarantee.* By following the procedure, as long as you tell the truth about your preferences, you will obtain what you feel is fair (even if everyone else lies). Thus there is an incentive to be truthful (and if you lie about your preferences, it can backfire).

A reader might object to the above particular cake-division solution, because the cutter only gets what he perceives to be 50 percent of the cake in his valuation, while the other person might end up with more. The solution is envy-free (neither person envies the other person’s share) but it is not *equitable*, meaning the perceived share of cake each player gets (in his own valuation) is different. This is not a deficit of the procedure (which only guaranteed envy-freeness and not equitability) as much as it is a fault in choice of procedure. An active area of research in mathematics [12], economics [7], and political science [3] is the development of fair division procedures in various settings and with various fairness criteria.

A more interesting fair division solution that has found application by practitioners is the Adjusted Winner procedure of Brams and Taylor [4]. It is a procedure

for dividing a set of goods between two parties in such a way that the division is: envy-free, equitable, and *efficient* (or *Pareto-optimal*). The last criterion means that there is no division that dominates the given one, i.e., there is no other division that is just as good for both parties and strictly better for one party. Thus the Adjusted Winner solution gives each party a share for which they do not wish to trade shares, and in which they feel they got just as good a portion as the other party feels it got, and there is no other solution that dominates. At most one of the goods may have to be divided in the procedure (though one cannot predict beforehand which good it is). We note that if there are more than two parties, it may not always be possible to satisfy these three properties in cake division, as discussed in [2].

The Adjusted Winner procedure has found application in divorce settlements [4] because of its fairness guarantees as well as its ease of use. In the procedure, both parties are given 100 points to divide by assigning over the objects. This is the part of the procedure where preferences are elicited. Then objects are initially given to the party that valued them most; such a division is efficient, but it may not be envy-free or equitable. Call the party who ends up with the largest fractional share (in its own evaluation) the *winner*, and the other, the *loser*. In the next phase of the procedure, the assignment of goods is “adjusted” by transferring goods from the winner to the other party in a particular order until both fractional shares are equalized.

The Adjusted Winner procedure has the 3 features described above for a fair division procedure. It has a *fairness guarantee*: what results is an outcome that is provably envy-free and efficient, in addition to being equitable. It relies on *mutilateral evaluation*: the preferences of both parties are taken into account, and the resulting division achieves the fairness guarantee for both parties using their own estimation. And it is *procedurally fair*: parties using the Adjusted Winner procedure can understand and verify the fairness guarantees for a particular solution (without having to understand the proofs); because they participated in the procedure by stating their preferences, they are more likely to feel that the outcome is fair.

As we shall see, these three fair division ideas can offer some helpful ideas to current thinking about redistricting which can be combined with other desired ideas for a good redistricting solution. They underlie the redistricting procedure of [6] that we will now explain. First, we will explain the problem of partisan unfairness that [6] attempts to address.

3. Redistricting: the problem of partisan unfairness

In most of the 50 states in the U.S., the districting protocol is to have one party draw all the boundaries. If the drawing party’s goal is to win as many districts as it can, the strategy is clear: draw boundaries so that each district either a) has a small majority of its voters, or b) has a large majority of the other party’s voters. In other words, for any district, the drawing party should strive to either win it by a small margin or lose it by a large margin (See [1] for a detailed discussion).

In general, with such a strategy, a drawing party with $X\%$ of support of the voters can win just under $\min(2X\%, 100\%)$ of the districts if there are no geometric constraints (e.g., requiring districts be compact or contiguous, etc.). In reality, geometric constraints usually mean that this ideal outcome cannot be achieved;

however, in most cases, the drawing party can still win a significantly larger percentage than $X\%$ of the districts, even with only partial knowledge of voting trends.

This is not just a theoretical issue, as has often been demonstrated when the party in control changes. We cite two examples:

- When Republicans took control of the Texas legislature in 2002, they redrew state districts mid-decade, and the Texas delegation changed from 15 Republicans and 17 Democrats to 22 Republicans and 10 Democrats. [8]
- In Michigan, the 2000 election produced 7 Republican representatives and 9 Democratic representatives. After the census, a new district map was drawn resulting in 9 Republican representatives and 6 Democratic representatives in the 2002 election (Michigan lost 1 seat due to the census). [8]

This ability of one party to draw districts in such a way as to gain political advantage is viewed as one of the major problems with redistricting in the United States; we shall refer to this as the problem of *partisan unfairness*. The districting protocol proposed in [6] avoids this inherent unfairness by ensuring that either party can win a percentage of districts that is very close to their fair share.

4. What is a party's fair share?

Defining a “fair” share is not as straightforward as it may seem. A reasonable first attempt would be to say that the percentage of districts won by a party should be close to the percentage of constituent voters in the party. However, geometric constraints can make this impossible. For instance, if one party enjoys a statewide 60% - 40% advantage in the electorate and if voters are mixed homogeneously throughout the state, then any reasonable district would have a similar (approximate) 60% - 40% advantage for the same party. In such a case, if we believe that districts should contain people who live in contiguous and relatively compact regions, then no matter how the boundaries were drawn, 100% of the districts would go to the majority party. We would consider this outcome fair (because of the geometry of how the voters are mixed) even though the outcome differed dramatically from our original (unachievable) notion of fairness: that only 60% of the districts should be won by the majority party. This example demonstrates the importance of geometric constraints in any reasonable notion of fair shares.

It may come as a surprise that a fairness notion can be defined, despite the complexity of possible geometric constraints. In this section, we define a fairness notion that incorporates geometric considerations such as how constituent voters are distributed and how districts are shaped. Before doing so, we make some preliminary definitions.

Any division of a state into districts that satisfies all desirable constraints (including legal constraints that district maps are subject to) will be called a *viable division*. The *voting outcome* V_{out} is a description of how every voter actually votes in a given election. Any party with a role in redistricting attempts to predict aspects of V_{out} to guide their choices of district boundaries. A *voting model* V shall be such a prediction of how every voter will vote.

A party's *rating system* R is an assignment of a number to any proposed viable division of a state that gives a measure of how desirable that viable division is. A simple example of a rating system is to rate a viable division by the number of districts the party thinks it can win; we will denote this rating system by R_{win} .

Implicit in calculating R_{win} is a voting model V that the party is using to predict which districts it will win.

In reality, a party's interests may be much more complicated than just the number of districts that it wins. A more general rating system is one where a party could rate the desirability of each district in a division (assigning it a number), then sum these numbers over all the districts to give a rating for the division. As in [6], we shall refer to such a rating system as *an additive rating system*, and denote any particular instance of it as R_{sum} . The rating system R_{win} is a special case of R_{sum} in which a party assigns a 1 to districts it expects to win and a 0 to those it expects to lose.

The more general rating system R_{sum} allows a party to take other considerations into account. Politically, these can be important, as the following examples demonstrate:

- perhaps some district has an incumbent who is on an important congressional committee, so winning that district is more valuable to the party (hence rated higher than other winnable districts),
- perhaps some district has an important landmark (a stadium or a construction project) worth more to a party than some other district,
- perhaps some district encompasses the supporters of two incumbents from the opposition party, so that even though the district will be lost, the elimination of one strong incumbent from the other party is valuable.

Equipped with the notions of viable division, voting model, and rating system, we can now define our fairness notion:

DEFINITION (Geometric Target). Given a voting model V and a rating system R , a party's *geometric target with respect to V and R* is the average of its highest and lowest ratings among the set of viable divisions. A party's geometric target with respect to V_{out} and R_{win} will be called the *absolute geometric target*.

The absolute geometric target is a definition of fairness that takes into account geometric constraints. There are several compelling reasons why this is a good definition. First, the definition seems conceptually fair as it lies exactly between the best and worst outcome (in terms of number of districts won) for each party. Second, when there are no geometric constraints, the absolute geometric target coincides with the percentage of constituent voters—as already mentioned, if the minority party has $X\%$ of the vote, its best outcome is to win about $2X\%$ of the districts, while its worst outcome is to lose all the districts 0% and so the absolute geometric target in this case would be approximately $\frac{2X+0}{2} = X\%$ of the districts. Third, because this definition uses only viable district maps, it incorporates geometric constraints by restricting attention to realizable outcomes. For instance, in the example at the beginning of this section (with the homogeneous $60\% - 40\%$ electorate split), both the best and worst rating for the minority party would be to win 0 districts which is the only possible outcome (and thus fair) in that case. We remark that the absolute geometric target could be used as a target for fairness for independent commissions.

The protocol for districting proposed in [6] (and described subsequently), allows each party the opportunity to achieve an outcome that is close to their own absolute geometric target. Moreover, it allows each party the opportunity to achieve an

outcome that is close to a geometric target with respect to any voting model V and any additive rating system R .

Note that the geometric target with respect to a voting model V and a rating system R is a notion that captures the fair division principle of *multilateral evaluation*: that party preferences should be taken into account. Each party has its own geometric target, based on its own voting model V and rating system R (derived from its preferences). This is to be distinguished from any absolute notions of fairness that might be imposed by an external arbiter (including the absolute geometric target).

We shall soon see that districting protocol of [6] will, in addition, possess the other features of a fair division procedure— *procedural fairness*, and a *fairness guarantee* (see Section 2).

5. The ranking protocol

Before presenting the redistricting protocol of [6], we present a simple but useful protocol, which we call the *ranking protocol*, for how two parties can decide on one of a bunch of outcomes. For our purposes, the setting will be that of two political parties, A and B , where the choice of outcomes are different proposed divisions of a state. The protocol is then simple: both parties rank the proposed divisions from best to worst from their perspective. Each proposed division then has two rankings. Select the proposed division whose worst ranking is best (this reflects the Rawls' maximin criterion, as discussed in [10]). If there are 2 such proposals (there can be at most 2), randomly choose one.

Notice that if there are n proposed divisions, the division chosen is guaranteed to be in the top $\frac{n}{2} + 1$ of both lists. Said another way, the ranking protocol provides an outcome that, for either party, is no worse than one less than their median outcome among the choices (and can be much better if the two parties desires are not diametrically opposed). As we shall see in the next section, the ranking protocol is used as an augmentation step for the core redistricting protocol of [6].

We point out more generally that the ranking protocol could be used with any proposed method for generating divisions of a state, be it divisions created by independent panels, computers, or based on any kind of optimization scheme.

6. The fair division redistricting protocol

We now describe the core protocol for redistricting that was presented in [6]. It will involve three parties: two parties with vested interest in the division (e.g. the democratic and republican parties, or the majority and minority party in a state) called parties A and B , and an independent agent which we'll refer to as I .

The formal precise description of the protocol can be found in [6]; here we will describe the protocol while working through an example that will illustrate many of the aspects of both the protocol and the resulting solution.

6.1. The State. Consider the map in Figure 1, which represents a state consisting of 25 parcels, which we are thinking of as indivisible units (here, they are rectangles or squares). Each parcel contains the same number of people; thus the smaller the parcel, the denser the population. Loosely, we can think of this state as having a city located at the small squares (T,U,V,W,X,Y), with suburban areas surrounding the city, and the remaining areas rural.

Suppose our goal is to divide this state into 5 districts, each containing exactly 5 parcels.

A		B		C		D	
E		F		G		H	
I	J	K		L		M	
		T	U	V	N		
O	P	W	X	Y	Q		
		R		S			

FIGURE 1. Sample map of districts.

Suppose that the voting outcome (V_{out}) of the ensuing election is given by Figure 2, which shows in each parcel the percentage of votes that party A receives.

		.53	.53	.59	.65		
		.59	.59	.65	.53		

FIGURE 2. Vote totals by district.

The ability to use the redistricting process to gain political advantage relies on the ability to predict some features of V_{out} . In general, of course, V_{out} is not known precisely at the time districts are being re-drawn. However, the combination of data from previous elections and opinion polling increasingly gives a more and more accurate model of how votes will be distributed. For the purposes of this example we will assume that the working voting model V of both parties coincides with V_{out} in Figure 2. In this example, party A has a slim statewide majority, receiving 50.12% of the total vote.

For this particular example, we will assume that the only thing the two parties care about is maximizing the number of districts they can win; thus their preferences are diametrically opposed with each having rating system R_{win} . We emphasize that this is an assumption we make for this example but that the protocol is designed

to work under much more general preferences—the additive ratings system R_{sum} , discussed earlier.

Notice that even though parties A and B each have approximately half the voters over the state, if either party is given complete control of the district-drawing process, they can draw districts so that they are the majority in 4 of the 5 districts. See Figure 3. In this example, the absolute geometric target for either party is $\frac{1+4}{2} = 2.5$ districts.

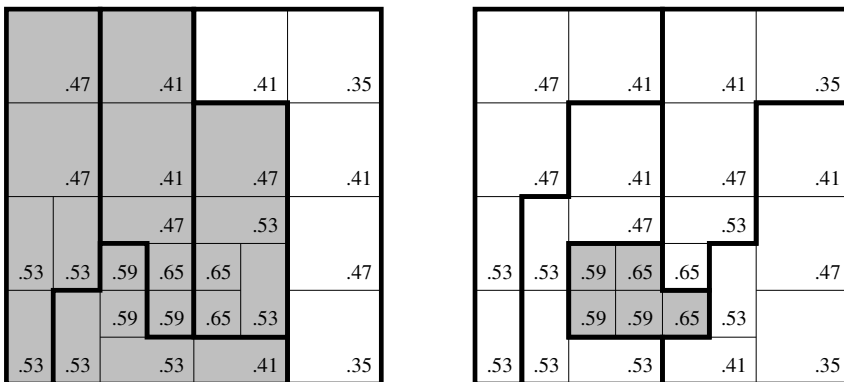


FIGURE 3. The left diagram shows a division in which A can win 4 districts. The right diagram shows a division in which B can win 4 districts. Districts that A wins are shaded.

6.2. The Protocol. There are three core steps to the redistricting protocol along with an augmenting fourth step. After outlining them in general, we will work through each step in the above example.

- **Split Sequence Generation.** This step is performed by the independent agent I . The agent generates a sequence of so-called k -splits: a k -split is a division of the state into two pieces (piece 1 and piece 2) such that the population within piece 1 totals the number of people in k districts. The independent agent I generates a *split sequence*: a 1-split, a 2-split, a 3-split, etc. with each split building on the previous so that piece 1 of the j -split contains piece 1 of the $(j - 1)$ -split for all j .
- **Preference.** For each of the k -splits, the two parties are each asked which of the following options they would prefer:
 - (1) to have party A divide piece 1 of the split into k districts and have party B divide piece 2 into $n - k$ districts (where n is the total number of districts).
 - (2) to have party B divide piece 1 of the split into k districts and have party A divide piece 2 into $n - k$ districts.

Each party has the option of saying that they are indifferent to the two choices².

- **Resolution.** If there exists an i -split such that parties A and B both prefer the same option in the preference step above then create a map using that option. If there exists an i -split such that one party is indifferent,

²This additional option is a modification of the original protocol in [6].

then create a map using the option selected by the party that was not indifferent. If there exists an i -split such that both parties are indifferent, then create a map by randomly choosing one of the options for that i -split.

If none of the above scenarios occur it means that the parties have opposite preferences for each i . Find the first i_0 , $1 \leq i_0 \leq n - 2$ for which party A prefers option (2) for $i = i_0$ and switches preferences to option (1) when $i = i_0 + 1$. (This scenario is guaranteed to occur at least once since party A prefers option (2) when $i = 1$ and prefers option (1) when $i = n - 1$.) Randomly choose to divide the state from the following four prescriptions:

- i. use option (1) for the i_0 -split,
 - ii. use option (2) for i_0 -split,
 - iii. use option (1) for $(i_0 + 1)$ -split,
 - iv. use option (2) for $(i_0 + 1)$ -split.
- **Augmentation.** We perform the above 3 steps for a number of different split sequences to produce a number of divisions of the state. We then use the ranking protocol of Section 5 to choose among these maps, i.e. each party ranks the divisions (from best to worst) according to their own preferences, and the division whose worst ranking (among both parties) is highest is the one that is chosen. If there are two such splits, select one of them at random.

6.3. The protocol in action. We now show how the protocol works for the example described in Figure 2.

Split Sequence Generation step. Suppose the Split Generation step yields the split sequence in Figure 4. In each diagram piece 1 will be the left piece and piece 2 will be the right piece.

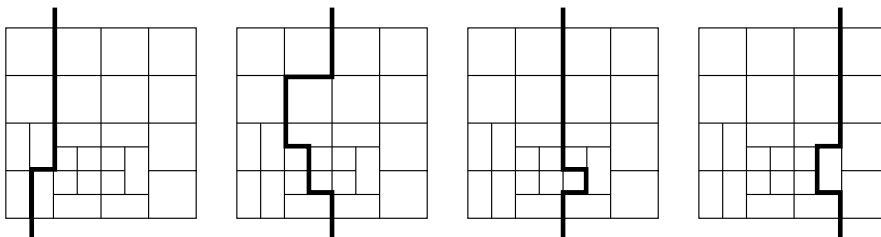


FIGURE 4. Split sequence generation. From left to right, a 1-split, 2-split, 3-split, 4-split.

Preference step. For the 1-split, we exhibit in Figure 5 a possible way each party can optimize its own interests in each of the two options. In option (1), A divides piece 1 and B divides piece 2. In option (2), B divides piece 1 and A divides piece 2.



FIGURE 5. Two options for the 1-split. The first diagram is option (1): in which A divides the left piece and B divides the right piece. The second diagram is option (2) in which B divides the left piece and A divides the right piece.

Thus party A would prefer option (2) (since it would win 4 out of 5 districts) and party B would prefer option (1) (since it would win 4 out of 5 districts). This is not surprising since there is no opportunity to gerrymander the left piece.

We then consider the same question for the 2-split. Figure 6 shows one possible way each party can optimize its own interests in each of the two options. Here, party A will still prefer option (2) and party B would still prefer option (1).



FIGURE 6. Two options for the 2-split. The first diagram is option (1): in which A divides the left piece and B divides the right piece. The second diagram is option (2) in which B divides the left piece and A divides the right piece.

For the 3-split, Figure 7 shows one possible way each party can optimize its own interests in each of the two options. Notice that now the parties preferences have changed: party A now prefers (1) and party B prefers (2).



FIGURE 7. Two options for the 3-split. The first diagram is option (1): in which A divides the left piece and B divides the right piece. The second diagram is option (2) in which B divides the left piece and A divides the right piece.

Finally, for the 4-split, Figure 8 shows one possible way each party can optimize its own interests in each of the two options. Again, party A prefers (1) and party B prefers (2).



FIGURE 8. Two options for the 4-split. The first diagram is option (1): in which A divides the left piece and B divides the right piece. The second diagram is option (2) in which B divides the left piece and A divides the right piece.

We summarize the results from party A 's point of view in the following table:

TABLE 1. The number of districts A wins on the left/right sides of split and total.

	option (1): left/right=total	option (2): left/right = total
1-split	1/1=2	1/3=4
2-split	1/1=2	1/2=3
3-split	3/0=3	1/1=2
4-split	4/0=4	1/0=1

With all the preferences now stated, we are ready to move to the final step of the protocol.

Resolution Step. Since the two parties prefer different options in each of the four splits, we find the point at which party A 's preference switches; here it occurs between the 2-split and the 3-split. Thus $i_0 = 2$ and we randomly choose between the four prescriptions corresponding to the options listed in the second and third row of Table 1.

- i. option (1) for the 2-split with the result: party A wins 2 districts, party B wins 3.
- ii. option (2) for the 2-split with the result: party A wins 3 districts, party B wins 2.
- iii. option (1) for the 3-split with the result: party A wins 3 districts, party B wins 2.
- iv. option (2) for the 3-split with the result: party A wins 2 districts, party B wins 3.

Notice that these results have party A winning either 40% or 60% of the districts, the two closest achievable percentages to both the percentage of votes for party A (50.12%) and the percentage of districts given by the absolute geometric target ($2.5/5 = 50\%$). This is the result that the protocol is designed to produce; it is argued in [6] that a rigorous result establishing a *good choice property* of the protocol combined with the way i_0 is chosen will result in this kind of behavior for most choice of split sequences. We discuss this in the next section.

Augmentation Step. For our example, we run the same protocol for the following four additional split sequences:

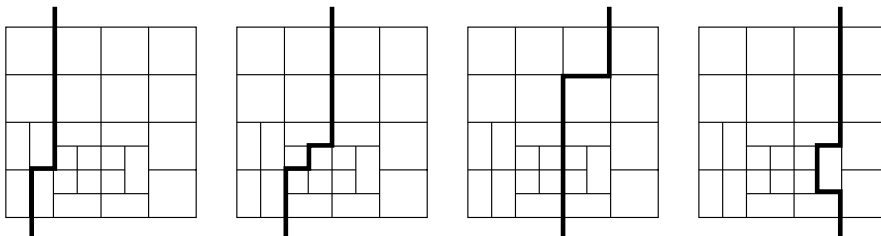


FIGURE 9. Vertical Split Sequence.

These have the following outcomes from party *A*'s perspective, listed in Table 2.

TABLE 2. The number of districts *A* wins in piece 1, piece 2, and total.

		option (1): pc.1/pc.2 = total	option (2): pc.1/pc.2 = total
Vertical Split	1-split	1/1=2	1/3=4
	2-split	1/1=2	1/2=3
	3-split	2/0=2	1/1=2
	4-split	4/0=4	1/0=1
Horizontal Split	1-split	0/1=1	0/4=4
	2-split	0/2=2	0/3=3
	3-split	2/1=3	1/2=3
	4-split	3/0=3	1/0=1
Diagonal Split 1	1-split	0/1=1	0/3=3
	2-split	1/1=2	0/3=3
	3-split	2/1=3	1/2=3
	4-split	3/0=3	1/0=1
Diagonal Split 2	1-split	1/1=2	1/3=4
	2-split	2/0=2	2/2=4
	3-split	3/0=3	2/1=3
	4-split	4/0=4	1/0=1

Unlike the first split sequence from Figure 4 that we explored in detail, the result of each of these split sequences is that the parties will be indifferent to one split. Here are possible maps from the results of the protocol for each of these split sequences:

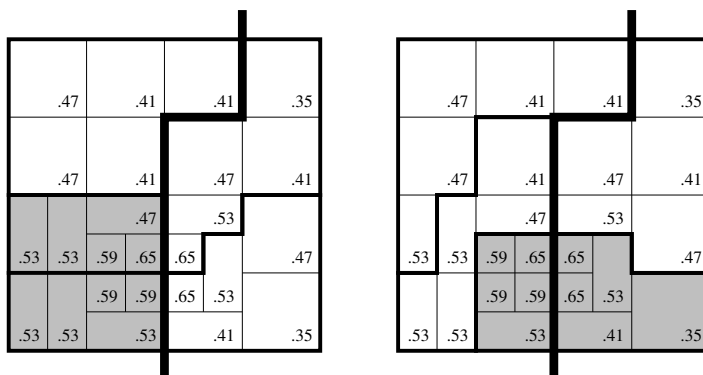


FIGURE 13. Outcomes for Vertical Split Sequence.

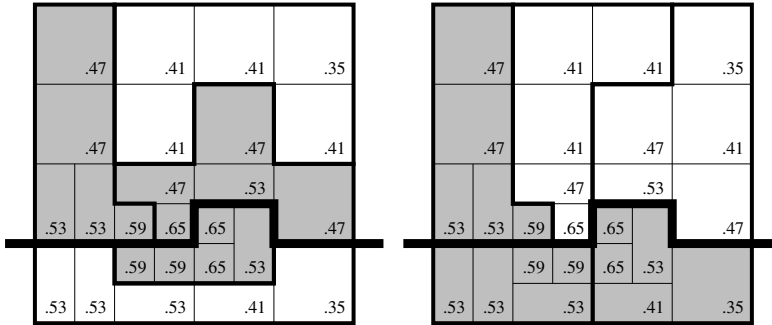


FIGURE 14. Outcomes for Horizontal Split Sequence.



FIGURE 15. Outcomes for Diagonal Split Sequence 1.

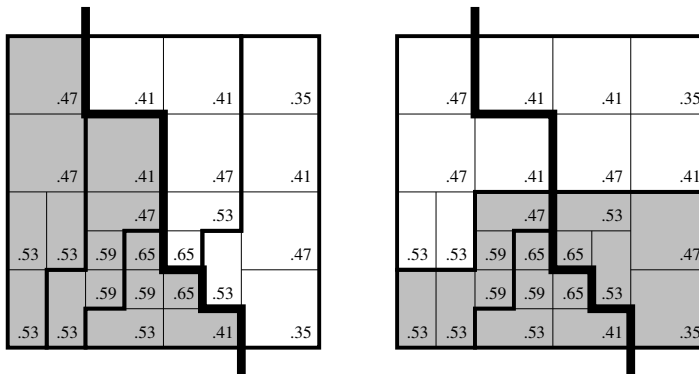


FIGURE 16. Outcomes for Diagonal Split Sequence 2.

In the ranking protocol, both parties would rank the 5 different outcomes from best to worst. Depending on the outcome of the random choice of prescription for the first split sequence (Figure 4), the outcomes would be party *A* winning 3 districts 3 or 4 times (Horizontal Split sequence, both Diagonal split sequences,

and possibly the first split sequence), and winning 2 districts 1 or 2 times (Vertical split sequence, and possibly the first split sequence). Since the result of the ranking protocol will result in one of the top three outcomes for both parties, in this case, the final outcome will be party A winning 3 districts and party B winning 2. We remark that this resolution is as close as one can get to both the absolute geometric target (2.5) and the proportion of constituent voters (50.12 %)

6.4. Fairness qualities of the protocol. Having explained the protocol we turn to a discussion of why the protocol is fair. We will analyze the protocol from the point of view of party A ; the identical analysis can be made for party B . We address the following two questions:

- *If the map is created from a choice party A preferred or was indifferent to, will it be fair for party A ?*
- *What if the randomness in the algorithm results in a choice party A did not prefer?*

The first question is answered in the affirmative by establishing that the protocol has the *good choice property* [6]. Approximately, the good choice property says that if party A is using a voting model V and has an additive rating system R , there will be a choice for party A that achieves an outcome that is at least as good as a number close to the geometric target for V and R (see Section 4 for definitions). Precisely, for a given k -split define *the party's k -split geometric target for V and R* to be the average rating of the best and worst outcomes over all viable divisions that include the dividing line of the k -split. Then the good choice property is:

THEOREM 6.1 (Good Choice Property [6]). *For any voting model V and rating system R , one of the choices given in the protocol achieves an outcome that is at least as good as the party's k -split geometric target for V and R .*

For our example above party A has been acting according to its interests with $V = V_{out}$ and $R = R_{win}$. The good choice property follows from the following observation that is perhaps best seen pictorially (see Figure 17): given a particular k -split, the average of the number of districts won by party A under options (1) and (2) is equal to the average of the number of districts that party A wins if it had complete control (which would result in the best outcome for party A) and if it had no control (which would result in an outcome no worse than the worst for party A).

Thus at least one of the two options is better than the average outcome between the best and worst scenario for party A , which is precisely the definition of the k -split geometric target when $V = V_{out}$ and $R = R_{win}$. It should be clear that the same argument holds regardless of choice of V and additive rating system R .

The astute reader will notice that the k -split geometric target for V and R can differ from the geometric target for V and R ; in other words, insisting that the division includes the boundary given by the k -split can penalize one party. Two observations suggest that should this happen, the penalty will not be large. First, the choice of split is made by an independent (neutral) third party and thus should be no more biased against one party than a random choice. Second, in the case where there are no geometric constraints (see [6]), the absolute geometric target and the k -split geometric target for V_{out} and R_{win} can differ by at most $\frac{1}{2}$. In our example, we see this difference between the k -split geometric target and the absolute geometric target: for party A , in the 3-split in Vertical Split sequence, and

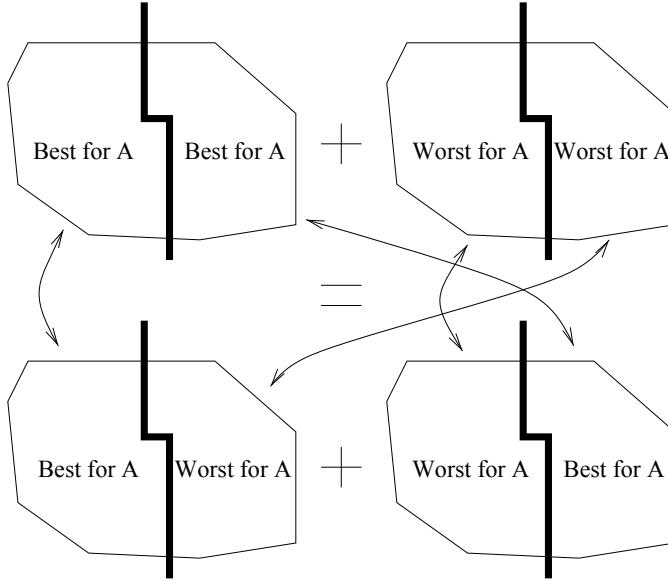


FIGURE 17. The good choice property.

for party B , in the 3-splits on the final three split sequences of the protocol. In each of these cases, however, the difference from the absolute geometric target is as small as it could be: $\frac{1}{2}$. It is reasonable to assume that most splits will either not particularly favor either party, or favor a party by a small amount. It is then the augmentation step of the protocol that ensures that a rare “bad” split for a particular party will not come into play (since the affected party would put such a split towards the bottom of their rankings).

We see therefore, that the good choice property, when coupled with the augmented protocol, implies that party A should be satisfied if the division is created by an option that it chose.

We now turn to the second question—how party A will fare if the randomness in the algorithm results in a choice they did not prefer. The randomness is implemented only if for each i -split, the two parties have opposite preferences (for instance in the first split sequence described in Figure 4). We suppose the random prescription in the Resolution Step (see Section 6.2) is one not preferred by party A , for instance prescription (i.) in the Resolution step (i.e. option (1) for $i = i_0$). In our example, this would correspond to $i_0 = 2$ in the first split sequence. Party A , however, preferred option (2): to divide piece 2 and have party B divide piece 1. Notice, however, that party A would prefer to divide up piece 1 in the $i_0 + 1$ split, and this piece 1 only differs from the piece 1 of the i_0 split by a small region with a population equal to the size of a single district. (Similarly piece 2 in the i_0 and $i_0 + 1$ splits only differ by this same small region). Because party A prefers option (2) for the i_0 split and option (1) for the $i_0 + 1$ split (and because piece 1 of these two splits do not differ by very much), it is reasonable to expect that party A ’s preference for option (2) over option (1) for the i_0 split is mild. (In our example, this is indeed the case as party A achieves an outcome of winning 2 districts which is of minimal negative deviation from the absolute geometric target of 2.5).

If indeed this is the case, then party A 's discontent with the division would only be mild as we have shown (by the good choice property) that party A would have been satisfied with the slightly better option of (2).

Even though the first pieces of the two splits differ by a small amount, one can construct scenarios where that small amount makes a big difference. However, recall that the creation of the splits was done by an independent party and therefore one would expect this type of scenario to be rare. Again, choosing to use the augmented protocol would ensure that this rare scenario would not occur in the division chosen.

7. Conclusion

Replacing current redistricting procedures with the protocol presented here surely presents substantial political obstacles. It has been observed numerous times (e.g., see [5]) that any proposed change should be structured to take effect far enough in the future so that it could not be interpreted as a power grab by one party. However, as noted in the last paragraph of the introduction, some of the ideas presented here could be incorporated into current processes without requiring a complete overhaul of the redistricting process.

In this article, we have used a detailed example to explore the redistricting protocol of [6]. We have shown how this procedure retains the usual constraints that may be desirable to impose on a redistricting solution, while incorporating some of the best features of a fair division procedure: multilateral evaluation, procedural fairness, and fairness guarantees. Procedural fairness is apparent in the protocol, the geometric targets incorporate multilateral evaluation, and the ability to ensure outcomes near geometric targets provides the fairness guarantee. The result is a solution that accounts for both parties having different interests, involves a resolution process and an interactive protocol to elicit preferences, and provides mathematical confidence that the outcome will be fair.

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DEPARTMENT OF ENGINEERING AND COMPUTER SCIENCE, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY CALIFORNIA 94720

E-mail address: zeph.landau@gmail.com

DEPARTMENT OF MATHEMATICS, HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA 91711

E-mail address: su@math.hmc.edu

When does approval voting make the “right choices”?

Steven J. Brams and D. Marc Kilgour

ABSTRACT. We assume that a voter approves a proposal depends on (i) the proposal’s probability of being right (or good or just) and (ii) the voter’s probability of making a correct judgment about its rightness (or wrongness). The state of a proposal (right or wrong), and the correctness of a voter’s judgment about it, are assumed, initially, to be independent. If the average probability that voters are correct in their judgments is greater than $\frac{1}{2}$, then the proposal with the greatest probability of being right will, in expectation, receive the greatest number of approval votes. This result also holds when voters’ probabilities of being correct are state dependent but not proposal dependent; when they are functionally related in a certain way; or when voters follow a leader with an above-average probability of correctly judging proposals. Sometimes, however, voters will more frequently select the right proposal by not following a leader and, instead, making their own independent judgments (as assumed by the Condorcet jury theorem). Applications of these results to different kinds of voting situations are discussed.

1. Introduction

Mahendra Prasad [17] recently proposed an extension of the Condorcet jury theorem (CJT) to voting on multiple proposals, wherein each proposal has a probability of being right, and each voter believes that every proposal is either right or wrong.¹ He argues that the proposal most likely to be right will be chosen by approval voting (AV). His paper includes both an overview of the classical political theory literature relating to normative social choice—especially the writings of Condorcet and Rousseau—and a survey of modern social choice theory that extends the CJT to multiple proposals.

In this paper, we assume that there are multiple proposals on a ballot; as in a referendum with several propositions that voters can support or oppose, more than one proposal can be approved. Although our world is black and white—a proposal is either right or wrong, and a voter’s judgment about it is either correct or incorrect—we embed it in a probabilistic framework, wherein each proposal has a probability of being right, and each voter has a probability of correctly judging its

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¹For “right” one can substitute desirable characteristics such as “just” or “good.”

state.² A proposal’s state, and a voter’s judgment about it, are assumed, initially, to be independent.

The paper proceeds as follows. In section 2, we prove that AV in expectation chooses those proposals mostly likely to be right if and only if the average probability that a voter is correct about the state of a proposal is greater than $\frac{1}{2}$ (Theorem 1).

In section 3, we assume that the probability that a voter is correct depends on a proposal’s state—whether it is right or wrong. We then show that AV chooses those proposals most likely to be right if and only if the *sum* of the average probabilities that a voter is correct about right and wrong proposals is greater than 1 (Theorem 2), which is a refinement of Theorem 1.

In section 4, we prove a negative result: AV does not always choose the proposal most likely to be right when the probability that a voter is correct depends on the proposal (Theorem 3). But in section 5 we show that if the average probability that a voter is correct, and the probability that a proposal is right, are functionally related in a certain way, the proposals that receive the most votes are most likely to be right (Theorem 4), echoing Theorems 1 and 2.

In section 6, we ask the following question: If all voters follow a leader who has an above-average probability of correctly judging whether a proposal is right, is their aggregated judgment better than when they vote independently? It turns out that it is—in the sense that AV better distinguishes right from wrong proposals—if the probability that proposals are right is never less than $\frac{1}{2}$ (Theorem 5). Surprisingly, however, voters who make independent judgments may have a greater probability of selecting the right proposals than following a leader, showing that different measures of the “rightness” of decisions can diverge (Theorem 6).

In section 7, we discuss applications of our results to different kinds of elections, pointing out that the deliberations of committees—including the one that debated US options in the 1962 Cuban missile crisis (EXCOM)—probably best approximate the use of AV. AV is also applicable to referendums with multiple propositions, wherein voters may approve of more than one.

In section 8, we relate our results to the CJT. The CJT concerns a single proposal and states that if (i) each voter has the same probability, greater than $\frac{1}{2}$, of being correct, and (ii) voters’ judgments of correctness are independent, then the probability that a majority of voters is correct approaches 1 as the number of jurors approaches infinity.³

Unlike the CJT, Theorems 1-6 do not posit a quota, such as a simple majority, but instead answer the question of which, among multiple proposals, are most likely to be right. We show under what conditions a proposal’s AV total can be interpreted as a measure of its probability of being right. We also consider the possibility of strategic voting.

In section 9, we summarize our results for juries that must weigh multiple charges or counts, legislatures that must decide among multiple bills or amendments to bills, and elections with multiple candidates. In these very different settings, the most approved choices tend to be those with the highest probabilities of being right.

²In contradistinction to Prasad [17], who assumes that proposals are either right or wrong with certainty, we assume that their rightness is probabilistic, making Prasad’s deterministic assumption a special case.

³For background and references on the CJT, see “Condorcet Jury Theorem,” [8]; additional references will be given in section 8.

2. Judging Multiple Proposals

We begin by computing the expected number of approval votes for a proposal when voters vote for all proposals that they judge to be right. Let $p(i)$ be the probability that proposal i is right, $i = 1, 2, \dots, m$, and let $q(j)$ be the probability that voter j judges a proposal correctly, $j = 1, 2, \dots, n$. We assume initially that a proposal’s being right, and a voter’s judgment of its being correct, are independent events, so $p(i)$ and $q(j)$ are unconditional probabilities. We also assume that these probabilities fall strictly between 0 and 1,

$$0 < p(i), q(j) < 1 \text{ for all } i \text{ and } j,$$

so no proposal is certain to be right or wrong, and no voter’s judgment is always correct or incorrect.

Assume that a voter approves of all proposals that he or she judges to be right and none that he or she judges to be wrong. These proposals might, for example, be different versions of a bill before a legislature or different charges in a criminal trial. We distinguish two cases: Exactly one proposal may be right, or the number of right proposals may be unrestricted. In the first case, the $p(i)$ ’s sum to 1, and in the second there is no restriction on their sum.

As an illustration of this distinction, assume that jurors in a criminal trial vote on multiple proposals (i.e., charges against a defendant; possible sentences). Effectively, they are using AV to provide a measure of support for each proposal.

We distinguish two scenarios:

- (1) Charges: A defendant is charged with reckless driving, drug possession, and assault, with the possibility of conviction or acquittal on each of the three separate charges (proposals). In this scenario, there are $2 \times 2 \times 2 = 8$ plausible vote combinations.
- (2) Sentences: There is a single charge of murder, and the jury has three choices: conviction for murder, conviction for manslaughter, or acquittal. Again, there are three choices, but not 8 plausible vote combinations.

In Sentences, a juror will approve of one or two (murder and manslaughter, if he or she most prefers murder, but presumably not if he or she most prefers manslaughter, because murder is a more serious charge with more dire consequences). In Charges, a juror will approve, or not approve, of each of the three charges.

In Charges, voting is akin to binary voting on propositions in a referendum; a juror will approve of the one option (the defendant is guilty or innocent) that he or she believes to be correct on each charge. By contrast, in Sentences, a voter may approve of more than one sentence (e.g., for murder and manslaughter). The one that receives the most approval will be the one that is implemented.

In Charges, a voter has no reason to be insincere, because there are only two options. Although our theorems apply to this scenario, other voting procedures, including plurality voting (PV), would give the same results.

By contrast, in Sentences, some voters may consider voting strategically. For example, if a juror believes that his or her preferred charge, murder, will not get the most votes but manslaughter may, that juror may well approve of both murder and manslaughter to try to ensure some conviction. Note that under PV, this juror would have to make a more drastic change—switching his or her vote from murder to manslaughter—and no longer would our theorems hold: PV may not yield the proposal most likely to be right.

These two scenarios illustrate how, in a criminal trial, a juror might be faced with more than two choices. Our subsequent analysis applies to voting under both scenarios, making it immaterial whether the $p(i)$'s sum to 1 or not.

There are two ways that voter j can decide to vote for proposal i : either (i) proposal i is right, and voter j judges it correctly, which has probability $p(i)q(j)$; or (ii) proposal i is wrong, and voter j judges it incorrectly, which has probability $[1 - p(i)][1 - q(j)]$. We wish to calculate the expected number of approval votes for proposal i , $AV(i)$, and the expected number of approval votes per voter, $av(i) = \frac{AV(i)}{n}$.

The theorem that follows depends on the average probability that a voter is correct about a proposal, $\tilde{q} = \frac{1}{n} \sum_{j=1}^n q(j)$. Note that, for now, this probability is assumed to be the same for all proposals. As we show next, if this average is high enough, then proposals that are more likely to be right are guaranteed to receive, in expectation, more approval votes.

THEOREM 2.1. *For any two proposals, i_1 and i_2 , the statement that*

$$av(i_1) > av(i_2) \text{ if and only if } p(i_1) > p(i_2)$$

is true if and only if $\tilde{q} > \frac{1}{2}$.

PROOF. As noted above, the probability that voter j votes for proposal i is $p(i)q(j) + [1 - p(i)][1 - q(j)]$. It follows that the expected number of approval votes by voter j for proposal i is also $p(i)q(j) + [1 - p(i)][1 - q(j)]$. Summing this expectation over all voters yields the expected number of approval votes received by proposal i :

$$(2.1) \quad AV(i) = \sum_{j=1}^n \{p(i)q(j) + [1 - p(i)][1 - q(j)]\}.$$

Multiplying out the summand and rearranging the terms of (2.1) yields

$$(2.2) \quad AV(i) = n - \sum_{j=1}^n q(j) + p(i) \left[2 \sum_{j=1}^n q(j) - n \right].$$

Dividing (2.2) by the number of voters, n , yields

$$(2.3) \quad av(i) = 1 - \tilde{q} + p(i) [2\tilde{q} - 1].$$

It follows from (2.3) that $av(i_1) - av(i_2) = [p(i_1) - p(i_2)] [2\tilde{q} - 1]$. Therefore, the signs of $av(i_1) - av(i_2)$ and $p(i_1) - p(i_2)$ are the same if and only if $2\tilde{q} - 1 > 0$, which is equivalent to $\tilde{q} > \frac{1}{2}$. Moreover, (2.3) shows that if $\tilde{q} = \frac{1}{2}$, then $av(i_1) = av(i_2)$ without regard to the values of $p(i_1)$ and $p(i_2)$, and that if $\tilde{q} < \frac{1}{2}$, the sign of $av(i_1) - av(i_2)$ is opposite to that of $p(i_1) - p(i_2)$. \square

As an illustration of Theorem 2.1, assume that jurors in a criminal trial vote on multiple proposals (i.e., charges against a defendant). Effectively, they are using AV to provide a measure of support for each charge. Because, as shown by (2.3), $av(i)$ is a strictly increasing function of $p(i)$ if $\tilde{q} > \frac{1}{2}$, the charge that receives the greatest support is the one most likely to be right.

Note that $AV(i) = n [av(i)]$ is the sum of n independent Bernoulli random variables (i.e., binomial random variables with one trial). Consequently, even though

$p(i_1) > p(i_2)$ and, as Theorem 2.1 guarantees, $av(i_1) > av(i_2)$, there is some probability that an unlikely event occurs and, for example, the actual vote for i_2 exceeds the actual vote for i_1 . However, by the law of large numbers, the probability of such a reversal approaches zero as the number of voters, n , becomes large. Moreover, this reversal probability diminishes as the gap between $av(i_1)$ and $av(i_2)$ increases, helping to ensure that the proposal most likely to be right is chosen by AV.

3. State Dependence

Assume next that the probability that voter j is correct about a proposal is not a constant, $q(j)$, but, instead, depends on whether the proposal is right or wrong:

- If proposal i is right, then voter j will judge it correctly with probability $q_r(j)$;
- If proposal i is wrong, then voter j will judge it correctly with probability $q_w(j)$.

Define $\tilde{q}_r = \frac{1}{n} \sum_{j=1}^n q_r(j)$ and $\tilde{q}_w = \frac{1}{n} \sum_{j=1}^n q_w(j)$ to be the average probabilities that a voter makes a correct judgment about, respectively, right and wrong proposals. This assumption of state dependence produces an extension of Theorem 2.1:

THEOREM 3.1. *For any two proposals, i_1 and i_2 , the statement that*

$$av(i_1) > av(i_2) \text{ if and only if } p(i_1) > p(i_2)$$

is true if and only if $\tilde{q}_r + \tilde{q}_w > 1$.

PROOF. We rewrite (2.1), replacing $q(j)$ by $q_r(j)$ if the proposal, i , being evaluated by voter j is right, and by $q_w(j)$ if this proposal is wrong:

$$AV(i) = \sum_{j=1}^n \{p(i)q_r(j) + [1 - p(i)][1 - q_w(j)]\}.$$

Proceeding as in the proof of Theorem 2.1 gives

$$(3.1) \quad AV(i) = n - n\tilde{q}_w + p(i) [n\tilde{q}_r + n\tilde{q}_w - n].$$

Dividing (3.1) by the number of voters, n , yields

$$(3.2) \quad av(i) = 1 - \tilde{q}_w + p(i) [\tilde{q}_r + \tilde{q}_w - 1].$$

Note that $av(i)$ is increasing in $p(i)$ if and only if $\tilde{q}_r + \tilde{q}_w > 1$. The remainder of the proof is analogous to that of Theorem 2.1. \square

Thus, the proposal that is most likely to be right receives, in expectation, the greatest number of approval votes, given that the *sum* of the average probabilities of being correct exceeds 1. Because $\tilde{q} = \frac{1}{2} [\tilde{q}_w + \tilde{q}_r]$, Theorem 3.1 reduces to Theorem 2.1 in the special case $\tilde{q}_w = \tilde{q}_r$, when the average probability that a voter’s judgment is correct does not depend on the state, because $\tilde{q} = \tilde{q}_w = \tilde{q}_r$.

4. Proposal Dependence

In section 2, we assumed that the probability that a voter correctly judges a proposal depends only on the voter and not on the proposal. In section 3, we assumed that this probability can depend on the proposal, but only insofar as the true state is right (r) or wrong (w). Thereby we replaced $q(j)$ with two probabilities,

$q_r(j)$ and $q_w(j)$, that were in general different. If, for example, voter j is better at correctly judging proposals that are right than those that are wrong, $q_r(j) > q_w(j)$.

In this section, we assume that a voter's ability to judge a proposal may be different for every proposal, even those in the same state of rightness or wrongness. Specifically, we assume that the probability that voter j 's judgment is correct about proposal i is $q(i, j)$, a function of i as well as j .

Let $Q(i) = \sum_{j=1}^n q(i, j)$. Then $\overline{q(i)} = \frac{1}{n}Q(i)$ is the average probability that a voter is correct about proposal i . Now—contra Theorems 2.1 and 3.1—we obtain a negative result about the correctness of the voters' judgments about competing proposals.

THEOREM 4.1. *If voters' probabilities of correct judgment are proposal dependent, then, even if $\overline{q(i)} > \frac{1}{2}$ for all values of i , it is possible for two proposals, i_1 and i_2 , to satisfy $av(i_1) < av(i_2)$ and $p(i_1) > p(i_2)$.*

PROOF. In (2.1), we replace $q(j)$ with $q(i, j)$ to obtain

$$AV(i) = \sum_{j=1}^n p(i)q(i, j) + [1 - p(i)][1 - q(i, j)]$$

Proceeding as before, we obtain

$$(4.1) \quad AV(i) = n - q(i) + p(i)[-n + 2q(i)]$$

where $q(i) = \sum_{j=1}^n q(i, j)$. Dividing (4.1) by the number of voters, n , gives

$$(4.2) \quad av(i) = 1 - \overline{q(i)} + p(i)[-1 + 2\overline{q(i)}]$$

where $\overline{q(i)} = \frac{1}{n}q(i)$.

Now we use (4.2) to show that it is possible for two proposals, i_1 and i_2 , to satisfy $p(i_1) > p(i_2)$ and $av(i_1) < av(i_2)$. Consider two proposals, $i = 1$ and 2, where $p(1) = 0.8$, $\overline{q(1)} = 0.55$, $p(2) = 0.7$, and $\overline{q(2)} = 0.75$. From (4.2),

$$av(1) = 1 - 0.55 + (0.8)[-1 + 2(0.55)] = 0.45 + (0.8)(0.1) = 0.53.$$

$$av(2) = 1 - 0.75 + (0.7)[-1 + 2(0.75)] = 0.25 + (0.7)(0.5) = 0.60.$$

Hence, even though $p(1) > p(2)$, $av(1) < av(2)$. □

Theorem 4.1 shows that a proposal more likely to be right (i_1) may receive, in expectation, fewer approval votes than a proposal less likely to be right (i_2). Thus, unlike in Theorems 2.1 and 3.1, Theorem 4.1 demonstrates that the expected number of approval votes for a proposal need not increase in its probability of being right. The reason is that both the second and the third terms on the right side of (4.2) depend on i ; in particular, the second term, $\overline{q(i)}$, is not a constant, as were the analogous terms, \bar{q} and $\overline{q_r}$, in (2.3) and (3.2), respectively.⁴

In addition, whereas the bracketed term in (4.2) causes $av(i)$ to increase as $p(i)$ increases if $\overline{q(i)} > \frac{1}{2}$, the other appearance of $\overline{q(i)}$ in (4.2) has a negative sign, and so has the opposite effect. Hence, $av(i)$ may not increase when $p(i)$ increases, as the preceding example demonstrates.

⁴While differentiation of (2.3) and (3.2) with respect to $p(i)$ would cause these terms to vanish, this is not the case for $\overline{q(i)}$ in (4.2) unless it is known that there is no functional relation between $\overline{q(i)}$ and $p(i)$.

Of course, if the average probability that a voter is correct does not depend on the proposal, that is, if $q(i) = \bar{q}$ for all i , then Theorem 2.1 applies. When this is the case, as we showed in section 2, there is a positive association between the approval votes for a proposal and the probability that it is right. Are there other kinds of dependence in which this positive association holds?

5. Other Kinds of Dependence

As we show next, there can be a positive association between the approval votes for a proposal and its probability of being right, even when the voters’ probabilities of judging correctly depend on the proposal. We now assume that any proposal has a probability p of being right and an average probability \bar{q} of being judged correctly by the voters. We will characterize such a proposal (p, \bar{q}) .

For illustration, assume that all proposals satisfy $\bar{q} = p$.⁵ In words, the probability that a proposal is judged correctly by all voters equals the probability that it is right. Then, by (4.2), the average approval vote of proposal (p, p) equals

$$(5.1) \quad av = 1 - p + p[-1 + 2p] = 2p^2 - 2p + 1$$

Differentiating (5.1) with respect to p yields

$$(5.2) \quad \frac{dav}{dp} = 4p - 2,$$

which is positive if and only if $p > \frac{1}{2}$, a condition that is analogous to $\bar{q} > \frac{1}{2}$ in Theorem 2.1, wherein \bar{q} was assumed constant.

Thus, if $\bar{q} = p$, approval votes track the probability that proposals are right when $p > \frac{1}{2}$, illustrating how the conclusions of Theorems 2.1 and 3.1 can hold even when the correctness of voters’ judgments is proposal-dependent. From (5.2) we know that an infinitesimal increase in p will produce an infinitesimal increase in av . More specifically, within the interval $\frac{1}{2} < p \leq 1$, any two proposals can be compared, and the one with the greater p will, in expectation, receive more approval votes.

We now generalize the foregoing example to all proposals that have a probability p of being correct. We assume that the probabilities are selected from the interval of real numbers, $[0, 1]$, and for every proposal, the average probability that the voters judge it correctly, \bar{q} , is related to p by a differentiable function, which we denote $\bar{q}(p)$.⁶ In Theorem 2.1, the function $\bar{q}(p)$ is constant (i.e., does not depend on p); in the example just discussed, $\bar{q}(p) = p$. Now we do not assume a particular functional form for $\bar{q}(p)$ but, rather, derive the conditions that it must satisfy to make the expected number of approval votes a strictly increasing function of p .

THEOREM 5.1. *Suppose that for all proposals (p, \bar{q}) , \bar{q} is functionally related to p and, moreover, $\bar{q}(p)$ is differentiable. If there is a subinterval of values of p where $f(p) = (2p - 1)(2\bar{q} - 1)$ is an increasing function of p then, whenever p_1 and p_2 lie in this subinterval and $p_1 > p_2$, AV chooses, in expectation, a proposal that is right with probability p_1 over one that is right with probability p_2 .*

⁵Because $p(i)$ and $\bar{q}(i)$ are assumed equal and, therefore, do not depend on i (their values depend only on each other), we can eliminate i as an argument.

⁶This assumption is in fact rather restrictive. It implies, for example, that any two proposals with the same value of p also have the same value of \bar{q} . Moreover, a small change in the value of p must correspond to a small change in the value of \bar{q} .

PROOF. Writing (4.2) without dependence on i , we differentiate av with respect to p :

$$(5.3) \quad \frac{dav}{dp} = -\frac{d\bar{q}}{dp} + p \left[2\frac{d\bar{q}}{dp} \right] + [-1 + 2\bar{q}] = \frac{d\bar{q}}{dp} [-1 + 2p] + [2\bar{q} - 1]$$

For comparison, $\frac{d}{dp}f(p) = 2 \left[[2\bar{q} - 1] + (2p - 1)\frac{d\bar{q}}{dp} \right]$. The theorem follows. \square

It is easy to link Theorem 5.1 with Theorem 2.1: When $\bar{q}(p) = \bar{q}$ is constant, $f(p) = (2p-1)(2\bar{q}-1)$ is an increasing function of p if and only if $\bar{q} > \frac{1}{2}$, in which case the subinterval is $[0, 1]$. In the case of $\bar{q}(p) = p$ discussed earlier, $f(p) = (2p-1)^2$, and it is easy to verify that the conditions of Theorem 5.1 are satisfied for the subinterval $[\frac{1}{2}, 1]$ —that is, if and only if $p \geq \frac{1}{2}$. A parallel example is $\bar{q}(p) = 1 - p$, in which case $f(p) = -(2p-1)^2$, so the subinterval is $[0, \frac{1}{2}]$.

Many other examples could be constructed. The most realistic, we think, are those in which $\bar{q}(p)$ is monotonically increasing in p , but not necessarily linearly. For example, $\bar{q}(p)$ may increase slowly near $p = \frac{1}{2}$, but then rapidly as p approaches 1, if the proposals most likely to be right are much more likely to be judged correctly. To aid in the construction of such examples, we note that Theorem 5.1 is equivalent to the conditions that $\frac{d\bar{q}}{dp} > \frac{1-2\bar{q}}{2p-1}$ whenever $p > \frac{1}{2}$; $\bar{q} > \frac{1}{2}$ when $p = \frac{1}{2}$; and $\frac{d\bar{q}}{dp} < \frac{1-2\bar{q}}{2p-1}$ whenever $p < \frac{1}{2}$.

To summarize, when voters' probabilities of being correct depend on the proposal being considered, AV does not necessarily single out the proposals most likely to be right (Theorem 4.1). However, if the average voter's probability of being correct is a differentiable function of the probability of the proposal's being right, then Theorem 5.1 provides a condition on this function that ensures that the expected number of approval votes of a proposal reflects the probability that that proposal is right.

6. Follow-the-Leader

For convenience, we henceforth assume that a voter's judgment is equally good—on any proposal, whether it is right or wrong—rendering Theorem 2.1 applicable. In expectation, therefore, the proposal that receives the most approval votes is the one with the greatest probability of being right if and only if $\bar{q} > \frac{1}{2}$.

We next ask whether voters might improve the chance that a proposal most likely to be right is selected if they all follow the advice of some leader, $j = L$. We denote by $av_L(i)$ the average number of approval votes received by proposal i when all voters follow L .

One might expect that follow-the-leader would be an especially good strategy for selecting the proposal most likely to be right when $q(L) > \bar{q}$, or L has an above-average probability of judging proposals correctly. The next theorem shows that this is indeed true—follow-the-leader surpasses the independent judgments of voters in distinguishing candidates with the greatest probabilities of being right, based on their AV totals. However, this result is complicated by an issue that we will discuss shortly.

THEOREM 6.1. *If $q(L) > \frac{1}{2}$, then for two proposals i_1 and i_2 , $av_L(i_1) > av_L(i_2)$ if and only if $p(i_1) > p(i_2)$. Moreover, $av_L(i_1) - av_L(i_2) > av(i_1) - av(i_2)$ if and only if $q(L) > \bar{q}$.*

PROOF. Replacing \bar{q} by $q(L)$ in (2.3)—because the average q for L is simply $q(L)$ when all voters follow his or her advice—gives the average number of approval votes that proposal i receives from a voter who votes according to L 's choice:

$$(6.1) \quad av_L(i) = 1 - p(i)[2q(L) - 1].$$

It follows from (6.1) that

$$av_L(i_1) - av_L(i_2) = [p(i_1) - p(i_2)][2q(L) - 1].$$

Thus, if $q(L) > \frac{1}{2}$, then because $p(i_1) > p(i_2)$, proposal i_1 receives, in expectation, more approval votes if all voters follow the leader, L .

But recall from (2.3) that

$$av(i_1) - av(i_2) = [p(i_1) - p(i_2)][2\bar{q} - 1].$$

Then

$$(6.2) \quad \frac{av_L(i_1) - av_L(i_2)}{av(i_1) - av(i_2)} = \frac{2q(L) - 1}{2\bar{q} - 1}.$$

It is now easy to verify that the fraction on the right side is greater than 1 if and only if $q(L) > \bar{q}$. \square

Thus, L must indeed be above average in his or her ability to judge proposals correctly to make it rational for voters to follow his or her advice rather than relying on their own independent judgments. In doing so, voters increase the chances that AV will choose the proposal(s) most likely to be right—by widening the difference between the approval vote totals of better and worse proposals—compared with what independent judgments produce.

More specifically, from (6.1) the gap between $av_L(i_1)$ and $av_L(i_2)$ is a positive multiple of $q(L) - \bar{q}$. This implies that it is beneficial to choose the best leader—that is, the person with the highest value of $q(L)$ —assuming this information is known.

But there is another issue, which we illustrate with a simple example with two voters, L and F (for follower), where $q(L) = 0.8$ and $q(F) = 0.6$ so that $\bar{q} = 0.7$. Assume there are two proposals, with $p(1) = 0.9$ and $p(2) = 0.8$.

If the voters exercise their judgments independently, then according to (2.3),

$$av(1) = 1 - 0.7 + (0.9)[2(0.7) - 1] = 0.3 - (0.9)(0.4) = 0.66$$

$$av(2) = 1 - 0.7 + (0.8)[2(0.7) - 1] = 0.3 - (0.8)(0.4) = 0.62.$$

On the other hand, if the voters follow the leader L , then according to (6.1),

$$av_L(1) = 1 - 0.8 + (0.9)[2(0.8) - 1] = 0.2 - (0.9)(0.6) = 0.74$$

$$av_L(2) = 1 - 0.8 + (0.8)[2(0.8) - 1] = 0.2 - (0.8)(0.6) = 0.68.$$

Notice that proposal 1 garners, in expectation, more approval votes than proposal 2, regardless of whether the voters make their own independent judgments or follow L . However, as guaranteed by Theorem 6.1, follow-the-leader provides a bigger “spread” between proposals 1 and 2 ($0.74 - 0.68 = 0.06$) than if L and F record their independent judgments ($0.66 - 0.62 = 0.04$). Thus, if the procedure were repeated many times, we would expect that the average vote for proposal 1 would be greater under follow-the-leader.

However, it is not true that follow-the-leader *more surely* chooses the proposal with the greater probability of being right. As we show next, the probability that follow-the-leader favors proposal 1 over proposal 2 is 0.237. On the other hand,

the probability that independent judgments favor proposal 1 is 0.331. Hence, the probability that follow-the-leader gives the correct decision is actually less than independent judgments.

How can this be? Using the foregoing example, we formulate this result next.

THEOREM 6.2. *The proposal that is most likely to be right can be chosen less frequently under follow-the-leader than under independent judgments.*

PROOF. Under follow-the-leader, proposal 1 beats proposal 2 if and only if L approves of proposal 1 and does not approve of proposal 2, a judgment we code as $(1,0)$. In fact, if L approves of proposal 1 and disapproves of proposal 2, so will F ; hence, proposal 1 will defeat proposal 2 by 2-0.

The leader, L , approves of proposal 1 when

- (1) L judges proposal 1 correctly (with probability 0.8), and proposal 1 is right (with probability 0.9), which has a joint probability of 0.72.
- (2) L judges proposal 1 incorrectly (with probability 0.2), and proposal 1 is wrong (with probability 0.1), which has a joint probability of 0.02.

These probabilities sum to 0.74. Similarly, L disapproves of proposal 2 when

- (1) L judges proposal 2 correctly (with probability 0.8), and proposal 2 is wrong (with probability 0.2), which has a joint probability of 0.16.
- (2) L judges proposal 2 incorrectly (with probability 0.2), and proposal 2 is right (with probability 0.8), which has a joint probability of 0.16.

These probabilities sum to 0.32.

It follows that the probability that L 's judgment is $(1,0)$ is the product, $0.74 \times 0.32 \approx 0.237$. Of course, in the follow-the-leader model, F follows L , so proposal 1 defeats proposal 2 under follow-the-leader with probability 0.237. By comparison, L 's judgment is $(1,1)$ with probability 0.503, $(0,1)$ with probability 0.177, and $(0,0)$ with probability 0.083. In the latter three cases, of course, proposal 1 does not defeat proposal 2 under follow-the-leader.

These possibilities exhaust the approval/disapproval choices of L , so their probabilities necessarily sum to 1. Note that the most likely event is $(1,1)$, occurring more than half the time (0.503), in which L (and F) approve of both proposals and thereby create a tie between them.

We next show, suppressing some details, that independent judgments give a higher probability that proposal 1 will defeat proposal 2 than does follow-the-leader, even though L is more likely to be right than F (0.8 vs. 0.6) and is therefore a better-than-average judge. Proposal 1 can defeat proposal 2 in three different ways, shown in the first column of the following table:

The leader (L) and follower (F) columns code the judgments of L and F that give rise to the vote combinations to the left. For example, $(2,1)$ occurs if both L and F approve of proposal 1 but only L approves of proposal 2. We have calculated the probabilities, in a manner analogous to that for L in the case of $(1,0)$ under follow-the-leader, in the fourth column for independent judgments. They sum to 0.331, which is 40 percent higher than the probability of 0.237 that we found for follow-the-leader.⁷ □

⁷The divergence between the probability that a proposal is right and its expected approval vote is mirrored in other realms. For example, the probability of a net gain from a sequence of bets, on the one hand, and the expected amount from that gain on the other, may be at odds. Davis ([9], pp. 23-29) gives the example of a seemingly profitable bet in which you win twice

Vote Combinations	Leader (L)	Follower (F)	Probability
(2,1)	(1,1)	(1,0)	0.128
(1,2)	(1,0)	(1,1)	0.077
(2,0)	(1,0)	(1,0)	0.060
(1,0)	(1,0)	(0,0)	0.044
(0,1)	(0,0)	(1,0)	0.021
Sum			0.331

TABLE 1. Vote Combinations and Probabilities of Being Right under Independent Judgments

Because in our example there are only two proposals and two voters—each with a high probability of correctly judging the proposals—there is a substantial probability of ties: 59 percent with follow-the-leader, 40 percent with independent judgments. Although we focused on the situation in which the proposal more likely to be right (proposal 1) receives strictly more support than the other proposal (proposal 2)—making proposal 1 the “winner”—in situations such as referendums with multiple propositions on the ballot, more than one proposition might win (e.g., with majority approval).

If the number of voters is large, ties become highly unlikely with independent judgments; with follow-the-leader, of course, they never occur, because all the voters vote alike, regardless of how many there are. While follow-the-leader and independent judgments will give a similar, if not the same, ranking of proposals, there is an important difference in how they determine rankings.

In comparing two proposals, follow-the-leader makes an error (judging a right proposal to be incorrect, judging a wrong proposal to be correct) whenever the leader makes an error; this error rate does not decrease as the number of voters increases. On the other hand, the error rate does decrease as number of voters increases in the case of independent judgments, and in general yields a number of approval votes very close to the expected number if the number of voters is large.

As long as there is some difference in the expected number of approval votes that different proposals receive, a large enough electorate will reliably distinguish better from worse proposals under both follow-the-leader and independent judgments. Whereas follow-the-leader is superior at drawing this distinction, its dependence on all-or-nothing votes gives it a higher error probability in selecting the right proposal(s). As with the Condorcet Jury Theorem (CJT)—to which we will later compare our results—having a large number of voters tends to produce the right outcome, however votes are aggregated or whatever the decision rule is for choosing a winner.

your bet with probability $\frac{1}{2}$ and lose just your bet with probability $\frac{1}{2}$. If you start with \$100 and always bet 75 percent of your capital, after only two bets your expected winnings are \$189.06, but 75 percent of the time you end up with less than \$100. To maximize the rate of growth of your capital, it turns out, you should bet a fraction chosen not to maximize your expected winnings, but rather their logarithm, which is known as Kelly betting. This is a so-called Markov betting system, because the outcome depends only on your present bankroll and your probability of winning ([10], pp. 60-62).

7. Applications to Politics

At the beginning of their deliberations in a case, jurors are often divided on a verdict. But, typically, they move toward a decision—unanimous or near-unanimous, as specified by the rules that govern the case—making hung juries relatively rare. On average, about 10 percent of all cases result in hung juries [13].

During their deliberations, jurors will often be persuaded by the juror who offers the most persuasive arguments, whom we assume has an above-average probability of being correct. Assume that this juror is L , and for definiteness assume that $q(L) = \max\{q(j) : j = 1, 2, \dots, n\}$ —that is, $q(L)$ is the maximum value of $q(j)$.

If all jurors follow L , we showed in section 6 that, provided $q(L) > \bar{q} > \frac{1}{2}$, the proposals with the greatest p 's—the ones most likely to be right—will in expectation garner the most approval votes. Nonetheless, the probability that a proposal with a greater value of p is selected may actually increase if the jurors exercise their own independent judgments.

This argument against following the lead of L is made by opponents of “group-think” [14], in which independent thinking is suppressed in favor of achieving a group consensus, often leading to poor decisions. On the other hand, if we assume that the average juror is genuinely persuaded by L , where $q(L) > \bar{q} > \frac{1}{2}$, independent thinking will not be suppressed but, instead, be replaced by the superior thinking of L , based on the more persuasive arguments L offers compared with those offered by other jurors.

Of course, if L 's arguments persuade jurors to support proposals that are more likely to be wrong than right (i.e., $p(i) < \frac{1}{2}$), then follow-the-leader will have a perverse effect. But this will not be true if $q(L) > \bar{q} > \frac{1}{2}$, in which case follow-the-leader will draw a sharper distinction than independent judgments between better and worse proposals, though independent judgments may maximize the probability that the better proposal will be chosen.

Our model is applicable to groups other than juries. As a case in point, consider the deliberations of EXCOM, the executive committee of high-level government and other officials who debated options that the United States might choose during the Cuban missile crisis of October 1962 ([2], pp. 226-240, and references therein). Although EXCOM members at the outset leaned toward an air strike against the Soviet missiles in Cuba, most of its members were persuaded in the end to recommend to President John Kennedy the less aggressive action of a naval blockade (called, euphemistically, a “quarantine” at the time) and only consider more aggressive action if the blockade failed to induce the Soviets to withdraw their missiles from Cuba.

In the deliberations of EXCOM, Robert Kennedy, the attorney general and brother of President Kennedy, seems to have fulfilled the role of L . He warned that an air strike would be seen as “a Pearl Harbor in reverse, and it would blacken the name of the United States in the pages of history” ([18], p. 684). To be sure, the fact that Robert Kennedy and his supporters were successful in persuading other EXCOM members to support a blockade cannot be taken as conclusive evidence that follow-the-leader will always succeed, but it does illustrate one instance in which persuasion seems to have abated a major political-military crisis, leading to its peaceful resolution.

In democracies, political parties and their candidates put forward proposals to solve problems and advance their positions; suppose that associated with each

proposal is a probability of its being right, or at least providing some remedy. Are the proposals selected (including the status quo) the ones most likely to be right?

Our model is inapplicable to legislatures and other voting bodies wherein proposals come up one at a time and then are voted up or down. Because voting is sequential in these bodies, voters cannot approve or disapprove, simultaneously, of multiple proposals. In such settings, the ordering of proposals (e.g., amendments to a bill) that are voted on can, for strategic reasons, critically affect the support they receive, so their votes do not provide an accurate gauge of their degree of sincere support.

In elections in which there are multiple candidates on a ballot, usually a choice of only one candidate is possible. Even if the voter is permitted to rank the candidates, this ranking does not say where the voter would draw the line between approved and disapproved candidates, though systems have been proposed that would allow this ([1], ch. 3; [7]). Besides juries that consider multiple charges, or committees like EXCOM that deliberate over multiple strategies, referendums with multiple propositions on the ballot come closest to fitting the AV model. The propositions can be considered proposals, and voters can approve of more than one.

Usually a simple majority determines which propositions pass. If, however, two or more propositions contradict each other, and each gets a majority of votes—as can happen—the usual rule is that the proposition with the most votes is enacted. Because this is the proposition most likely to be right according to our model, this rule is consistent with passage of those (noncontradictory) propositions most likely to be right.⁸

In both jury/committee settings and referendums, voters typically follow different leaders, who may espouse different positions. The question our analysis raises is whether the leader who persuades the most voters to approve of his or her favored proposal helps the proposal most likely to be right.

Because there is not usually a single L but, instead, multiple leaders who take different positions on proposals, one must be careful how to define “right.” Previously, we defined $p(i)$ to be the probability that proposal i is right (or good or just).

But suppose that there are two leaders, one of whom supports proposal i and the other of whom opposes it. Assume that all voters support the positions of one of the two leaders. Then if we interpret $p(i)$ to be the probability that the supporter of proposal i is right, and $1 - p(i)$ to be the probability the opponent is right, then AV will choose the proposal with the higher probability of being right.

While this interpretation of our model certainly applies to multiple propositions in a referendum⁹—in which one can approve or not approve of each—how does it apply to elections with multiple candidates? We suggest that a useful way to think about candidates who take positions on multiple proposals is as *composites of positions*. Under AV, the voter who approves of one or more candidates is saying,

⁸For analyses of different ways of aggregating votes in referendums with multiple propositions, see [5] and [6].

⁹In a referendum, there is, of course, a third option—namely, to abstain. If there is no quorum, abstention has no effect, but if a minimum percentage (e.g., 50) of the electorate must participate to allow for the passage of a proposition, then if this minimum is not achieved, it seems reasonable to interpret nonenactment as the right choice, even if the proposition receives majority support of those who participated. This is because the failure to achieve a quorum can be deemed as insufficient support to make a choice binding on the electorate.

in effect, that he or she approves of their composite positions—at least more so than the composite positions of other candidates that fail to receive his or her approval.

In this interpretation, the $p(i)$'s are associated with each candidate i , who represents a composite of positions on what we earlier called proposals—the issues of the day in an election. But are the candidates who receive the most approval the ones whose composites of positions are the ones most likely to be right?

In the context of elections, “appealing” might be a better word to use than “right,” because there is usually no right or wrong position, or composite of positions, as such (unlike the guilt or innocence of a defendant in a criminal trial). But if we associate the appeal or popularity of a candidate with his or her being the right choice, then AV will make the right choice in elections.

To be sure, the “people’s choice” in such elections is not what many political philosophers, at least since Plato, would consider the right choice. But if the popular will—even if it does not always mirror the ideal of Rousseau’s general will—is the cornerstone of democracy, then it is appropriate to consider it synonymous with the right choice in elections.¹⁰

8. Relationship to the Condorcet Jury Theorem (CJT)

The CJT assumes that there is a single proposal, which is either right or wrong. Unlike our model, it does not have a probability associated with being in one state or the other.

Like our model, however, a proposal’s rightness or wrongness is judged by jurors who themselves have probabilities of being correct. The CJT says that if all jurors have the same probability, greater than $\frac{1}{2}$, of being correct, then the proposal’s probability of being judged correctly by a majority of jurors approaches 1 as the number of jurors approaches infinity.¹¹ By contrast, Theorem 2.1 does not assume that every voter j has a probability of being correct, or $q(j)$ that exceeds $\frac{1}{2}$, but, instead, only that the mean of all voters’ probabilities, \bar{q} , exceeds $\frac{1}{2}$, in order that the approval votes for proposals mirror their probabilities of being right.¹²

Unlike the CJT, majority rule has no special significance: Except when there is proposal dependence (Theorem 4.1), the proposal that receives the most approval votes, which may or may not be a majority, is the one most likely to be right. But as with CJT, as the number of voters increases, the probability of making the right choice increases, provided that the added voters do not reduce the value of \bar{q} .¹³

Might another voting system choose the proposal or proposals most likely to be right? Consider plurality voting, in which each voter is restricted to casting one

¹⁰Miller [15] makes this argument in applying the Condorcet jury theorem to elections, but with the qualifying phrase that all voters, even if they have conflicting interests, are “fully informed.” But if voters can be seduced (as opposed to persuaded by logic and reason) by populist or demagogic appeals, then it is dubious to equate appeal with rightness.

¹¹Nitzan [16] (pp. 205-207) shows under rather general conditions when simple-majority rule gives a higher probability that a proposal is judged correctly than the “expert rule,” which is follow-the-leader when the leader is the voter with the greatest probability of being correct.

¹²An “extended” CJT ensures that if an average juror has a probability of being correct that is greater than $\frac{1}{2}$, the probability that a jury will make the right decision approaches a value that is a function of e , and is strictly less than 1 [12].

¹³This is true without regard to the values of the $p(i)$'s, which may happen, for example, if a company will surely fail if it does nothing. However, if there is some less-than-even chance of success if it takes some risky action, then the most approved action, even if it will probably fail, is still better than doing nothing.

vote. In order for him or her to select the proposal most likely to be right, a voter must be able to identify the degrees of rightness of each proposal i , as given by $p(i)$, in order to choose the one most likely to be right. But we assume that the $p(i)$'s are unknown to the voters; in the absence of this knowledge, the aggregation of plurality votes need not single out the proposal most likely to be right, even if \bar{q} is high.

To get plurality voting to choose the proposal most likely to be right, voters' judgments about proposals would need to be conditioned on each proposal's rightness. But even for approval votes, as we showed in Theorem 4.1, this creates problems. Only when the voters' average probability of judging a proposal correctly is functionally related (in an appropriate manner) to the probability that the proposal is right (Theorem 5.1) can the approval votes of proposals reflect their probabilities of being right.

Our model assumes that voters respond to a signal—based on $q(j)$ or perhaps $q(L)$ —that they receive on each proposal i ; this proposal has a probability, $p(i)$, of being right. Might voters do better responding strategically rather than sincerely?

To inquire about strategy presumes that voters have preferences over outcomes, which the $q(j)$'s in our model do not assume. However, if one makes this assumption (see [11] and references therein), jurors can do better by conditioning their decisions on their probabilities of being pivotal, which will depend on both the decision rule and how other jurors vote. Thus, for example, if a verdict requires unanimity in a criminal trial, then a juror will be pivotal if and only if all the other jurors vote to convict or acquit, making his or her vote decisive either in convicting or in acquitting the defendant.

But when there are more than a few voters, a voter's pivotalness becomes less meaningful as a basis for making a choice. Indeed, the voter's probability of being decisive becomes negligible as the number of voters becomes larger and larger. Moreover, under AV, the question is less one of making the right choice on a single proposal and more one of where to draw the line between acceptable and unacceptable proposals, as analyzed in [3] and [4].

In the present model, voters seem well advised to make their own best judgments about proposals, either according to $q(j)$ or by following a leader according to $q(L)$. To deviate from these signals, voters—or the leaders whom they follow—would need to have information, which we do not assume, that there is at least the potential to produce more right choices by ignoring or countermanding their signals. Unless the strategic environment provides voters with the opportunity to obtain this information, it seems reasonable to assume that they will be sincere.

True, if voters follow different leaders, then the strategic situation changes—a competitive election is no longer just a search for right choices. For example, a leader may advise a voter not to vote for a candidate for whom the voter receives a favorable signal, lest this candidate beat a candidate preferred by the leader. On the other hand, if multiple proposals can be approved, as in a referendum, and supporting one proposal does not affect the choice of another, strategic voting is not an issue.

9. Conclusions

We have shown that the most approved proposals will be those with the greatest probability of being right if and only if the average probability that the judgment

of a voter is correct exceeds $\frac{1}{2}$ (Theorem 2.1). This necessary and sufficient condition allows some voters to have probabilities of being correct that are less than $\frac{1}{2}$, provided they are counterbalanced by voters who raise the average above $\frac{1}{2}$.

Although Theorem 2.1 and the subsequent theorems bear some similarity to the CJT, their differences are substantial. First, except for Theorems 4.1 and 5.1, the theorems assume that proposals have probabilities of being right that are independent of the judgments of voters. Second, there is not a single proposal but multiple proposals, all of which may have varying degrees of rightness.

While the most approved proposals under AV will be the ones most likely to be right in most circumstances, this may not true under plurality voting. The reason is that a voter, not knowing the $p(i)$'s, must cast his or her single vote on the basis of his or her $q(j)$ alone, which does not distinguish among proposals. Under AV, however, all voters vote with some positive probability for all the proposals; except for Theorem 4.1, our theorems ensure that the expected number of approval votes is greater for the proposals more likely to be right.

More specifically, this is true not only if proposal probabilities and voter probabilities are based on independent events but also if the probability that a voter makes a correct judgment about a proposal depends on its state (i.e., whether it is right or wrong, as shown in Theorem 3.1). While this is not generally true if voter probabilities depend upon the proposal being considered (Theorem 4.1), approval votes track the rightness of proposals if the average probability that a voter is correct, and the probability that a proposal is right, are functionally related in certain ways (Theorem 5.1).

So does follow-the-leader, if the leader has an above-average probability of being correct, which sometimes—but not always—may be preferable to voters' making independent judgments (Theorem 6.1). This may be one reason why defendants, who think their case is strong, sometimes prefer that their case be heard by a judge with a high $q(j)$ than a jury with a lower \bar{q} . However, when the number of voters is small, as in a committee, the independent judgments of its members may more often lead to the right decision (Theorem 6.2), illustrating the divergence between the probability that a collective choice is right and its expected approval vote.

AV is most applicable to situations in which there are multiple alternatives that voters must choose among, such as criminal charges in a trial, proposals in a committee, or candidates in an election, all of which have some probability of being right (i.e., $p(i) > 0$). We have shown that AV is well suited to finding the best—the most likely to be right, good, or just—among them, although strategic considerations may interfere if there are multiple leaders contesting elections.

We conclude on a note of caution. Our results on selecting the proposals most likely to be right—except for calculating the probability that a 2-person committee makes the right decision in section 6—are based on the expected approval of these proposals, which will not always be realized in practice. Especially if the electorate is small, random variability may occasionally imply that the most approved proposals are not be the ones most likely to be right. As the electorate increases in size, however, the correctness of choices becomes more and more certain under AV—without the need, à la the CJT, to assume that every voter is better than a random coin toss.

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DEPARTMENT OF POLITICS, NEW YORK UNIVERSITY, NEW YORK, NEW YORK
E-mail address: `steven.brams@nyu.edu`

DEPARTMENT OF MATHEMATICS, WILFRID LAURIER UNIVERSITY, WATERLOO, ONTARIO,
 CANADA
E-mail address: `mkilgour@wlu.ca`

How indeterminate is sequential majority voting? A judgement aggregation perspective

Klaus Nehring and Marcus Pivato

ABSTRACT. We consider the degree of indeterminacy which can arise in a judgement aggregation problem when propositions are decided one at a time through majority vote, with earlier decisions placing logical constraints on later decisions. The outcome of such “sequential majority voting” can be highly path-dependent. A judgement aggregation problem is “globally indeterminate” if the truth value of *every* proposition is path-dependent. We show that, for several common classes of judgement aggregation problems, such global indeterminacy is not only possible, but fairly commonplace. We then consider problems which are “fully indeterminate,” meaning that every possible assignment of truth-values can arise from some path.

1. Introduction

Consider a group of voters who must collectively decide the truth values of some finite set of propositions. If the propositions were logically independent, then their truth values could be decided separately through majority vote. However, if the propositions are logically interconnected, then majority vote will often lead to a logically inconsistent outcome [Con85, Gui52, KS86, LP02, NP10]. This observation has motivated two lines of inquiry: first, to understand how pervasive and severe this problem of collective logical inconsistency can be, and second, to find some way to aggregate the views of the voters in a logically consistent manner. These two lines of inquiry form the subject of *judgement aggregation* [LP09, LP10, Mon12].

We will refer to a system of logically interconnected propositions as an *aggregation space*, and an assignment of truth values to the propositions as a *view*.¹ Each voter has a logically consistent view; the problem is to obtain a consistent collective view. As we have noted, the “majority view” may be inconsistent. One way to resolve this problem is to look for a view which is “as majoritarian as possible,” while still respecting the constraint of logical consistency. In other words, if we cannot

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¹We will define these terms more precisely later.

have a logically consistent view which agrees with the majority on *all* propositions, then we seek a logically consistent view which agrees with the majority in a maximal *subset* of propositions. The set of all views with this property is called the *Condorcet set*.

Another way to resolve the problem of majoritarian inconsistency is for the voters to apply majority vote to the different propositions one at a time, but with the recognition that earlier decisions act as logical constraints on later decisions. This can be seen as an abstract description of how many actually existing policy frameworks have evolved over time through a process of accretion. Such *sequential majority voting* is logically consistent by construction. But depending upon the *order* in which the propositions are decided, the group may converge on different outcomes. In other words, this process may be subject to *path-dependence*.

In two previous papers, jointly with Clemens Puppe, the authors have investigated the structure of the Condorcet set and the nature and extent of path-dependence [NPP14, NPP13]. These two issues are closely related, because it turns out that the Condorcet set is precisely the set of views which can be reached through sequential majority voting. The paper [NPP14] showed that many aggregation spaces exhibit a particularly thoroughgoing form of path-dependence, called *global indeterminacy*; this means that the truth value of *every* proposition depends on the path taken.

The present paper extends this exploration. Section 2 introduces notation and terminology. Section 3 shows that the global indeterminacy identified in [NPP14] is not an improbable or “fluke” occurrence; it arises naturally in aggregation spaces which are large enough or have enough internal symmetries. Furthermore, for many important aggregation spaces, it is in fact quite a robust phenomenon, which can be generated even with a very small population of voters. Section 4 explores the most extreme possible form of path-dependence, *full indeterminacy*. This means that *any* possible view can arise as the outcome of some path. Section 5 focuses on a particular necessary condition for such full indeterminacy. Full indeterminacy is relatively rare. But Section 6 shows that many classes of aggregation spaces are “almost” fully indeterminate, in the sense that they asymptotically approach full indeterminacy as the number of propositions becomes large. All proofs are in the Appendix.

2. Preliminaries

Let \mathcal{K} be a finite set of propositions or issues. An element $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$ is called a *view*, and interpreted as an assignment of a truth value of ‘true’ (1) or ‘false’ (0) to each proposition in \mathcal{K} .² Not all views are feasible, because there will be logical constraints between the propositions (determined by the structure of the underlying decision problem faced by society). Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ be the set of ‘admissible’ or *consistent* views; we call \mathcal{X} an *aggregation space*.

EXAMPLE 2.1. (a) (*Preferences*) Let $\mathcal{A} = \{a, b, c\}$ be a set of three alternatives. We can represent the space of all preference orders over \mathcal{A} with the following aggregation space. Let $\mathcal{K} := \{“a \succ b”, “b \succ c”, “c \succ a”\}$, and let $\mathcal{X} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$. Here, for example, $(1, 1, 0)$ is the view

²What we call a ‘view’ is conceptually equivalent to what other papers called a ‘judgment set’ [LP09] or a ‘binary evaluation’ [DH10].

that says $a \succ b$ and $b \succ c$, but not $c \succ a$ —in other words, the preference order $a \succ b \succ c$. The other five elements of \mathcal{X} correspond to the other five preference orders over \mathcal{A} .

(b) (*Committees*) Suppose we must select a committee of exactly two people, and the candidates for this committee are Alice, Bob, Chiara, Daoud, and Elise. Let $\mathcal{K} = \{A, B, C, D, E\}$, and let $\mathcal{X} = \{(1, 1, 0, 0, 0), (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (1, 0, 0, 0, 1), (0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 1, 0, 0, 1), (0, 0, 1, 1, 0), (0, 0, 1, 0, 1), (0, 0, 0, 1, 1)\}$. Here, for example, the view $(1, 0, 1, 0, 0)$ represents the committee consisting of Alice and Chiara.

(c) (*Choice along on a line*) Suppose we must select some value $x \in \{0, 1, 2, 3, 4\}$. Let $\mathcal{K} = \{“x \geq 1”, “x \geq 2”, “x \geq 3”, “x \geq 4”\}$, and let $\mathcal{X} = \{(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)\}$. Here, for example, the view $(1, 1, 0, 0)$ asserts that $x \geq 1$, and $x \geq 2$, but $x \not\geq 3$, and $x \not\geq 4$ —in other words, $x = 2$.

(d) (*Truth functions*) Let p, q , and r be three logical propositions, and let $s = p \& q \& r$. Suppose we must assign truth-values to these four propositions in a logically consistent way. Thus, $\mathcal{K} = \{p, q, r, s\}$ and $\mathcal{X} = \{(0, 0, 0, 0), (0, 0, 1, 0), (0, 1, 0, 0), (0, 1, 1, 0), (1, 0, 0, 0), (1, 0, 1, 0), (1, 1, 0, 0), (1, 1, 1, 1)\}$. \diamond

An *anonymous profile* is a probability measure on \mathcal{X} —that is, a function $\mu : \mathcal{X} \rightarrow [0, 1]$ such that $\sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x}) = 1$ with the interpretation that, for all $\mathbf{x} \in \mathcal{X}$, $\mu(\mathbf{x})$ is the proportion of the voters who hold the view \mathbf{x} . For any $\mathcal{Y} \subseteq \mathcal{X}$, define $\mu(\mathcal{Y}) := \sum_{\mathbf{y} \in \mathcal{Y}} \mu(\mathbf{y})$. Let $\Delta(\mathcal{X})$ be the set of all anonymous profiles. For any $\mu \in \Delta(\mathcal{X})$, and any $k \in \mathcal{K}$, let

$$\tilde{\mu}_k := \mu\{\mathbf{x} \in \mathcal{X} ; x_k = 1\}$$

be the total ‘popular support’ for the position “ $x_k = 1$ ”.

EXAMPLE 2.2. Let \mathcal{X} be the three-alternative preference aggregation space from Example 2.1(a) above. Suppose $\mu(1, 1, 0) = 0.31$, $\mu(1, 0, 1) = 0.29$, $\mu(0, 1, 1) = 0.25$ and $\mu(0, 0, 1) = 0.15$. In other words, 31% of the voters assert that $a \succ b \succ c$, 29% assert that $c \succ a \succ b$, 25% assert that $b \succ c \succ a$, and 15% assert that $c \succ b \succ a$. Thus $\tilde{\mu}_{a \succ b} = 0.31 + 0.29 = 0.6$, $\tilde{\mu}_{b \succ c} = 0.31 + 0.25 = 0.56$, and $\tilde{\mu}_{c \succ a} = 0.29 + 0.25 + 0.15 = 0.69$. \diamond

Let $\Delta^*(\mathcal{X}) := \{\mu \in \Delta(\mathcal{X}); \tilde{\mu}_k \neq \frac{1}{2}, \forall k \in \mathcal{K}\}$ be the set of anonymous profiles where there is a strict majority supporting either 0 or 1 in each coordinate.³ For any $\mu \in \Delta^*(\mathcal{X})$, the *majority view* $\text{Maj}(\mu) \in \{0, 1\}^{\mathcal{K}}$ is defined as follows: for any $k \in \mathcal{K}$, define $\text{Maj}_k(\mu) := 1$ if $\tilde{\mu}_k > \frac{1}{2}$, and $\text{Maj}_k(\mu) := 0$ if $\tilde{\mu}_k < \frac{1}{2}$. (For instance, if \mathcal{X} and μ are as in Example 2.2 above, then $\text{Maj}(\mu) = (1, 1, 1)$. Note that $(1, 1, 1) \notin \mathcal{X}$.)

It is quite common to find that $\text{Maj}(\mu) \notin \mathcal{X}$ [KS86, LP02]. So we will try to agree with the majority view in as many propositions as possible. Formally, for any $\mathbf{x} \in \mathcal{X}$, let $\mathcal{M}(\mathbf{x}, \mu) := \{k \in \mathcal{K} ; x_k = \text{Maj}_k(\mu)\}$ be the set of propositions in which \mathbf{x} coincides with the majority view. A “majoritarian” rule would presumably choose a view $\mathbf{x} \in \mathcal{X}$ which makes $\mathcal{M}(\mathbf{x}, \mu)$ as large as possible —a view which agrees with a maximal collection of majorities (on individual propositions) while remaining logically consistent. In the setting of preference aggregation (where the

³Usually, judgement aggregation is considered on *all* of $\Delta(\mathcal{X})$. However, our goal in this paper is to investigate the multiplicity of solutions in the Condorcet set of a generic profile; thus, we exclude the ‘nongeneric’ multiplicities which arise when $\tilde{\mu}_k = \frac{1}{2}$ for some $k \in \mathcal{K}$ —i.e. we confine our attention to profiles in $\Delta^*(\mathcal{X})$. (If the set of voters is large (respectively odd), then a profile in $\Delta(\mathcal{X}) \setminus \Delta^*(\mathcal{X})$ is highly unlikely (resp. impossible) anyways.)

propositions encode the orderings between pairs of alternatives), this principle was first advocated by Condorcet [Con85]. Thus, we will say that an element $\mathbf{x} \in \mathcal{X}$ is *Condorcet admissible* if there does not exist any $\mathbf{y} \in \mathcal{X}$ such that $\mathcal{M}(\mathbf{x}, \mu) \subsetneq \mathcal{M}(\mathbf{y}, \mu)$. Let $\text{Cond}(\mathcal{X}, \mu) \subseteq \mathcal{X}$ be the set of Condorcet admissible elements; we call this the *Condorcet set*. For instance, if \mathcal{X} and μ are as in Example 2.2 above, so that $\text{Maj}(\mu) = (1, 1, 1)$, then $\text{Cond}(\mathcal{X}, \mu) = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, which corresponds to the three preference orders $a \succ b \succ c$, $c \succ a \succ b$, and $b \succ c \succ a$.

For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{0, 1\}^{\mathcal{K}}$, say that \mathbf{y} is *between* \mathbf{x} and \mathbf{z} if, for all $k \in \mathcal{K}$, $(x_k = z_k = 0) \implies (y_k = 0)$ and $(x_k = z_k = 1) \implies (y_k = 1)$. Furthermore, \mathbf{y} is said to be *properly between* \mathbf{x} and \mathbf{z} if, in addition, $\mathbf{x} \neq \mathbf{y} \neq \mathbf{z}$. (For example, $(1, 1, 0)$ is properly between $(0, 1, 0)$ and $(1, 1, 1)$.) For any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$, write $\mathbf{x} \succ \mathbf{z}$ if there exists no $\mathbf{y} \in \mathcal{X}$ which is properly between \mathbf{x} and \mathbf{z} . (For example, if \mathcal{X} is the preference aggregation space from Example 2.1(a), then $(1, 1, 0) \succ (1, 1, 1)$, whereas $(0, 1, 0) \not\succeq (1, 1, 1)$.) The following simple observation is Lemma 1.5 of [NPP14].

LEMMA 2.3. **(a)** If $\text{Maj}(\mu) \in \mathcal{X}$, then $\text{Cond}(\mathcal{X}, \mu) = \{\text{Maj}(\mu)\}$.
(b) Otherwise, $\text{Cond}(\mathcal{X}, \mu) = \{\mathbf{x} \in \mathcal{X} ; \mathbf{x} \succ \text{Maj}(\mu)\}$. In this case, $|\text{Cond}(\mathcal{X}, \mu)| \geq 3$.

For many interesting aggregation spaces, the Condorcet set will be multivalued for some profiles. To make this precise, we must introduce some notation. Let $\mathcal{J} \subseteq \mathcal{K}$ and consider an element $\mathbf{w} \in \{0, 1\}^{\mathcal{J}}$, which corresponds to a subset of judgements on the issues in \mathcal{J} . The set \mathcal{J} is the *support* of \mathbf{w} , denoted $\text{supp}(\mathbf{w})$. We define $|\mathbf{w}| := |\mathcal{J}|$ to be the *size* of \mathbf{w} . For any $\mathcal{I} \subseteq \mathcal{J}$ define $\mathbf{w}_{\mathcal{I}} := (w_i)_{i \in \mathcal{I}}$, an element of $\{0, 1\}^{\mathcal{I}}$. For any $\mathbf{v} \in \{0, 1\}^{\mathcal{I}}$, we say \mathbf{v} is a *fragment* of \mathbf{w} (and write $\mathbf{v} \sqsubseteq \mathbf{w}$) if $\mathbf{w}_{\mathcal{I}} = \mathbf{v}$ —that is, if $v_i = w_i$ for all $i \in \mathcal{I}$. We say \mathbf{w} itself is a *forbidden fragment* for \mathcal{X} if, for all $\mathbf{x} \in \mathcal{X}$, we have $\mathbf{x}_{\mathcal{J}} \neq \mathbf{w}$. Finally, \mathbf{w} is a *critical fragment* if it is a minimal forbidden fragment—that is, \mathbf{w} is forbidden, and there exists no proper subfragment $\mathbf{v} \sqsubset \mathbf{w}$ such that \mathbf{v} is forbidden.⁴

EXAMPLE 2.4. (a) Let \mathcal{X} and \mathcal{K} be as in Example 2.1(a). Then the only two forbidden fragments are $(1, 1, 1)$ and $(0, 0, 0)$ (corresponding to the intransitive binary relations $a \succ b \succ c \succ a$ and $a \prec b \prec c \prec a$). These fragments are both critical.

(b) Let \mathcal{X} and \mathcal{K} be as in Example 2.1(b). Then a fragment is forbidden if it either has a “1” in more than two coordinates or a “0” in more than three coordinates. The critical fragments are those which either have a “1” in exactly three coordinates or a “0” in exactly four coordinates—that is, all permutations of the fragments $(1, 1, 1, *, *)$ and $(0, 0, 0, 0, *)$ (where “*” denotes an unspecified entry). \diamond

Let $W(\mathcal{X})$ be the set of critical fragments for \mathcal{X} , and let $\kappa(\mathcal{X}) := \max\{|\mathbf{w}| ; \mathbf{w} \in W(\mathcal{X})\}$. We say that \mathcal{X} is a *median space* if $\kappa(\mathcal{X}) = 2$. This means that all logical interrelations are confined to *simple* implications: for some $j, k \in \mathcal{K}$ and all $\mathbf{x} \in \mathcal{X}$, $x_j = 0$ implies that $x_k = 0$, or $x_j = 0$ implies that $x_k = 1$.

⁴Critical fragments have been introduced into social choice theory in [NP07, NP10] under the name ‘critical families’. They are called ‘minimal infeasible partial evaluations (MIPEs)’ in [DH10].

EXAMPLE 2.5. (a) Let \mathcal{X} and \mathcal{K} be as in Example 2.1(c). Then the critical fragments are $(0, 1, *, *)$, $(0, *, 1, *)$, $(0, *, *, 1)$, $(*, 0, 1, *)$, $(*, 0, *, 1)$, and $(*, *, 0, 1)$, because each of these corresponds to the (contradictory) assertion that x is both greater and less than some value. (For example, $(0, *, 1, *)$ asserts that $x < 1$ but $x \geq 3$, which is impossible.) All these critical fragments have size 2, so this space is a median space.

(b) Let \mathcal{X} and \mathcal{K} be as in Example 2.1(d). Then $(1, 1, 1, 0)$ is a critical fragment of size 4. Thus, this space is *not* a median space. Likewise, the spaces in Examples 2.1(a,b), are not median spaces, since they each contain critical fragments of size 3, as explained in Example 2.4. \diamond

We say that a space \mathcal{X} is *majority determinate* if $\text{Maj}(\mu) \in \mathcal{X}$ for all $\mu \in \Delta^*(\mathcal{X})$. Reformulating Fact 3.4 of [NP07], Proposition 1.9 of [NPP14] states that

Cond(\mathcal{X}, μ) is a singleton for all $\mu \in \Delta^*(\mathcal{X})$ if and only if \mathcal{X} is majority determinate, which in turn holds if and only if \mathcal{X} is a median space.

For instance, the Condorcet set is always a singleton for the linear aggregation space discussed in Example 2.5(a); in other words, majority voting always yields logically consistent outcomes for linear aggregation problems. Quite a few interesting aggregation spaces are median spaces. Familiar examples are lines (like Example 2.5(a)), trees (an obvious generalization of Example 2.5(a)), and distributive lattices [BM81, BM88, BJ91].⁵ One important example which is neither a tree nor a lattice is the space of “single-peaked” preferences on a linearly arranged set of alternatives; in this setting, the above fact reduces to Black’s well-known result concerning the existence of a Condorcet winner [Bla48].

However, many interesting aggregation spaces are *not* median spaces (e.g. Examples 2.1(a), (b), and (d)). Thus, the Condorcet set will often be multivalued. This can be interpreted as a form of path-dependence, as we now explain.

Let $\mathbf{y} \in \{0, 1\}^{\mathcal{K}}$. For any $i \in \mathcal{K} \setminus \mathcal{J}$, say that y_i is \mathcal{X} -consistent with $\mathbf{y}_{\mathcal{J}}$ if there exists some $\mathbf{x} \in \mathcal{X}$ with $\mathbf{x}_{\mathcal{J}} = \mathbf{y}_{\mathcal{J}}$ and $x_i = y_i$ —otherwise y_i is \mathcal{X} -inconsistent with $\mathbf{y}_{\mathcal{J}}$. Let $K := |\mathcal{K}|$. A *path* through \mathcal{K} is a bijection $\zeta : [1\dots K] \rightarrow \mathcal{K}$. We now define the ζ -sequential majority rule $F^\zeta : \Delta^*(\mathcal{X}) \rightarrow \mathcal{X}$. Let $\mu \in \Delta^*(\mathcal{X})$ and let $\mathbf{z} := \text{Maj}(\mu)$. Define $\mathbf{y} := F^\zeta(\mu) \in \mathcal{X}$ inductively as follows:

- Define $y_{\zeta(1)} := z_{\zeta(1)}$.
- Inductively, let $\mathcal{J} := \{\zeta(1), \zeta(2), \dots, \zeta(n)\}$, and suppose we have already decided $\mathbf{y}_{\mathcal{J}}$. Let $i := \zeta(n+1)$. If z_i is \mathcal{X} -consistent with $\mathbf{y}_{\mathcal{J}}$, then set $y_i := z_i$. Otherwise, set $y_i = \neg z_i$.⁶

Sequential majority voting is closely related to the Condorcet set because of the following observation [NPP14, Prop.1.8]:⁷

$$(2.1) \quad \text{Cond}(\mathcal{X}, \mu) = \{F^\zeta(\mu); \zeta : [1\dots K] \rightarrow \mathcal{K} \text{ some path}\}.$$

⁵For the beautiful theory of median spaces from the perspective of combinatorial geometry, see [BH83] or [vdV93].

⁶“ \neg ” represents logical negation. That is: $\neg 1 := 0$ and $\neg 0 := 1$.

⁷This can be seen as a judgment aggregation counterpart to Miller’s characterization of the top cycle as the outcome of non-strategic voting in simple binary tree agendas [Mil77].

Thus, the size of $\text{Cond}(\mathcal{X}, \mu)$ is a measure of the degree of *path-dependence* in sequential majority voting.⁸ Sequential majority voting describes the historical process through which decisions are often made. Thus, path-dependence can be problematic in at least two ways. First, suppose the path ζ is random and exogenous (e.g. the decisions on individual propositions in \mathcal{K} are made on an *ad hoc* basis, in response to political or legal exigencies of random origin). If μ is path-dependent, then the ultimate social choice $F^\zeta(\mu)$ will be, to some extent, random and arbitrary. Second, suppose the path ζ is chosen by an ‘agenda setter’ (e.g. the chairperson of a committee). If μ is path-dependent, then the agenda setter can choose ζ strategically, so as to manipulate the outcome $F^\zeta(\mu)$.

To further illustrate these concepts, consider classical Arrovian preference aggregation. Let \mathcal{N} be a finite set of social alternatives, and let \mathcal{K} be a subset of $\mathcal{N} \times \mathcal{N}$ which contains exactly *one* element of the set $\{(n, m), (m, n)\}$ for each distinct $n, m \in \mathcal{N}$. Then $\{0, 1\}^{\mathcal{K}}$ represents the space of all *tournaments* (i.e. complete, irreflexive, antisymmetric and possibly intransitive binary relations, or equivalently, complete directed graphs) on \mathcal{N} . Let $\mathcal{X}_{\mathcal{N}}^{\text{pr}} \subset \{0, 1\}^{\mathcal{K}}$ be the set of all tournaments representing total orderings (i.e. permutations) of \mathcal{N} . This space is often called the *permutahedron*. For instance, the space in Example 2.1(a) was the permutahedron on a set of three alternatives a, b, c . Classical Arrovian aggregation of strict preference orderings is simply judgement aggregation on $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$. For any profile $\mu \in \Delta^*(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$, the set $\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)$ is the set of preference orderings on \mathcal{N} such that no other ordering agrees with the μ -majority on a larger set of pairwise comparisons.

For any $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$, let $\overset{\mathbf{x}}{\prec}$ be the (possibly intransitive) binary relation on \mathcal{N} defined by \mathbf{x} . Moreover, for any $\mu \in \Delta^*(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$, let $\overset{\mu}{\prec}$ be the binary relation defined by $\text{Maj}(\mu)$ —the so-called *majority tournament*. An element $\mathbf{c} \in \mathcal{X}_{\mathcal{N}}^{\text{pr}}$ is called a *directed Hamiltonian chain* of $\text{Maj}(\mu)$ if all nearest-neighbour orderings in $\overset{\mathbf{c}}{\prec}$ agree with the orderings specified by $\overset{\mu}{\prec}$. In other words, if we represent $\overset{\mathbf{c}}{\prec}$ as a linear directed graph \mathcal{C} and represent $\overset{\mu}{\prec}$ as a complete directed graph \mathcal{D} in the obvious way, then \mathcal{C} is a (directed) subgraph of \mathcal{D} .

Let $\overset{\mathbf{x}}{\prec}_*$ be the transitive closure of $\overset{\mathbf{x}}{\prec}$, augmented by all pairs (n, n) for $n \in \mathcal{N}$; then $\overset{\mathbf{x}}{\prec}_*$ is a weak order (i.e. it is complete, reflexive and transitive). Let $\overset{\mathbf{x}}{\approx}_*$ be the symmetric part of $\overset{\mathbf{x}}{\prec}_*$. Then $\overset{\mathbf{x}}{\approx}_*$ is an equivalence relation, and the $\overset{\mathbf{x}}{\approx}_*$ -equivalence classes of \mathcal{N} are linearly ordered by the asymmetric part $\overset{\mathbf{x}}{\prec}_*$ of $\overset{\mathbf{x}}{\prec}_*$. The maximal $\overset{\mathbf{x}}{\approx}_*$ -equivalence classes of \mathcal{N} is called the *top cycle* of the tournament defined by \mathbf{x} . The next result completely characterizes the Condorcet set in the setting of preference aggregation.

PROPOSITION 2.6. *Let $\mu \in \Delta(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$.*

- (a) $\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu) = \{\mathbf{x} \in \mathcal{X}_{\mathcal{N}}^{\text{pr}}; \overset{\mathbf{x}}{\prec} \text{ is a Hamiltonian chain in } \overset{\mu}{\prec}\}$.
- (b) *For all $n, m \in \mathcal{N}$, we have $n \overset{\mu}{\prec}_* m$ if and only if $n \overset{\mathbf{x}}{\prec} m$ for all $\mathbf{x} \in \text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)$.*

Proof: See Proposition 3.1(a,b) of [NPP14]. □

⁸See [Lis04] and [DL07] for earlier investigations of path-dependence in majoritarian judgement aggregation.

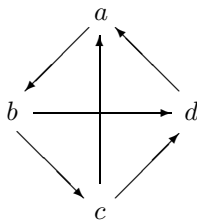


FIGURE 1. A majority tournament on four alternatives

As a consequence, the top cycle of the profile μ consists of all (and only) those alternatives which are ranked first by some preference order in $\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)$.

For example, consider the 4-permutahedron with alternatives a, b, c, d . Suppose that one third of the population endorses each of the preference orderings $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$ and $c \succ d \succ a \succ b$. For the corresponding majority tournament we have $c \succ^{\mu} a$, $d \succ^{\mu} a$, $a \succ^{\mu} b$, $b \succ^{\mu} c$, $b \succ^{\mu} d$, and $c \succ^{\mu} d$ (see Figure 1). By Proposition 2.6(a), the Condorcet set consists of the following five orderings: $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$, $c \succ d \succ a \succ b$, $d \succ a \succ b \succ c$, $c \succ a \succ b \succ d$. (In this case, the top cycle is $\{a, b, c, d\}$.)

3. Global indeterminacy

Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ be an aggregation space. We say that a profile $\mu \in \Delta^*(\mathcal{X})$ is *globally indeterminate* if, for any $k \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \text{Cond}(\mathcal{X}, \mu)$ such that $x_k \neq y_k$. Thus, for every proposition in \mathcal{K} , *either* truth value can arise from sequential majority voting for a suitably chosen path. The space \mathcal{X} is *globally indeterminate* if $\Delta^*(\mathcal{X})$ contains some globally indeterminate profiles.

For a simple example, consider preference aggregation. Let \mathcal{N} be a set of social alternatives, and let $\mu \in \Delta^*(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$ be a profile of preferences, such that every element of \mathcal{N} is in the top cycle (for example, the profile shown in Figure 1). Then Proposition 2.6(b) implies that μ is globally indeterminate: for any $a, b \in \mathcal{N}$, there exists some preference order in $\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)$ which prefers a to b , and there exists another preference order in $\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)$ which prefers b to a .

Let $W_3(\mathcal{X})$ be the set of critical fragments for \mathcal{X} of size 3 or more. Let $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$; we say that \mathbf{x} is *critical for \mathcal{X}* if there exists a collection $\{\mathbf{w}_1, \dots, \mathbf{w}_N\} \subseteq W_3(\mathcal{X})$ such that $\mathbf{w}_n \subseteq \mathbf{x}$ for all $n \in [1 \dots N]$ and $\mathcal{K} = \bigcup_{n=1}^N \text{supp}(\mathbf{w}_n)$ —in other words, \mathbf{x} can be completely “covered” by critical fragments of size 3 or more. Let $\text{Crit}(\mathcal{X}) := \{\mathbf{x} \in \{0, 1\}^{\mathcal{K}}; \mathbf{x} \text{ is critical for } \mathcal{X}\}$. Theorem 2.2 of [NPP14] is the following simple combinatorial characterization of global indeterminacy: a profile is globally indeterminate if and only if the majority view at this profile is critical for \mathcal{X} .

THEOREM 3.1. *Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$. For any $\mu \in \Delta^*(\mathcal{X})$,*

- (a) $(\mu \text{ is globally indeterminate}) \iff (\text{Maj}(\mu) \in \text{Crit}(\mathcal{X}))$.
- (b) *Let $\text{Maj}(\mathcal{X}) := \{\text{Maj}(\mu); \mu \in \Delta^*(\mathcal{X})\}$. Then*
 $(\mathcal{X} \text{ is globally indeterminate}) \iff (\text{Maj}(\mathcal{X}) \cap \text{Crit}(\mathcal{X}) \neq \emptyset)$.

EXAMPLE 3.2. Let \mathcal{X} be the “committee” space from Example 2.1(b). The critical fragments for this space are identified in Example 2.4(b). From these, it is easy to see that $\text{Crit}(\mathcal{X}) = \{(0, 0, 0, 0, 0), (1, 1, 1, 1, 1)\}$. Let $\mu \in \Delta^*(\mathcal{X})$ be the profile where each of the ten views in \mathcal{X} is supported by one tenth of the voters. Then $\text{Maj}(\mu) = (0, 0, 0, 0, 0)$. Thus, μ is globally indeterminate.

3.1. Global indeterminacy due to symmetry. Global indeterminacy can arise from the symmetries of the aggregation space. Let $\gamma : \mathcal{K} \rightarrow \mathcal{K}$ be a permutation. Define the bijection $\gamma_* : \{0, 1\}^{\mathcal{K}} \rightarrow \{0, 1\}^{\mathcal{K}}$ by

$$(3.1) \quad \gamma_*(\mathbf{x})_k := x_{\gamma(k)}, \quad \text{for all } \mathbf{x} \in \mathcal{X} \text{ and } k \in \mathcal{K}.$$

Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ be an aggregation space. We say that γ is a *symmetry* of \mathcal{X} if $\gamma_*[\mathcal{X}] = \mathcal{X}$. For example, *every* permutation of \mathcal{K} is a symmetry of the committee space from Example 2.1(b). For another example: let $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$ be as in Section 2, let $\tau : \mathcal{N} \rightarrow \mathcal{N}$ be any permutation of \mathcal{N} , and define $\gamma : \mathcal{K} \rightarrow \mathcal{K}$ by $\gamma(n, m) = (\tau(n), \tau(m))$ for all $n, m \in \mathcal{N}$; then γ is a symmetry of $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$ (and every symmetry of $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$ arises in this fashion).

Let $\Gamma_{\mathcal{X}}$ be the group of all symmetries of \mathcal{X} . For any $k \in \mathcal{K}$, the $\Gamma_{\mathcal{X}}$ -orbit of k is the set $\{\gamma(k); \gamma \in \Gamma_{\mathcal{X}}\}$. The set \mathcal{K} is a disjoint union of $\Gamma_{\mathcal{X}}$ orbits —call them $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_N$. (If \mathcal{X} is the committee space from Example 2.1(b), or $\mathcal{X} = \mathcal{X}_{\mathcal{N}}^{\text{pr}}$, then all of \mathcal{K} is a single $\Gamma_{\mathcal{X}}$ -orbit.) Let $K_n := |\mathcal{K}_n|$ for all $n \in [1 \dots N]$.

An element $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$ is $\Gamma_{\mathcal{X}}$ -fixed if $\gamma(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$. This is the case if and only if \mathbf{z} is constant on each of $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_N$. (Thus, there are exactly 2^N such $\Gamma_{\mathcal{X}}$ -fixed points in $\{0, 1\}^{\mathcal{K}}$.) Theorem 3.1 has the following consequence:

PROPOSITION 3.3. *Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$. Suppose there exists a $\Gamma_{\mathcal{X}}$ -fixed point $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$ with the following properties:*

- (a) *For all $n \in [1 \dots N]$, the fragment $\mathbf{z}_{\mathcal{K}_n}$ is forbidden to \mathcal{X} .*
- (b) *There exists some $\mathbf{x} \in \mathcal{X}$ such that $\#\{k \in \mathcal{K}_n; x_k = z_k\} > |\mathcal{K}_n|/2$ for all $n \in [1 \dots N]$.*

Then \mathcal{X} is globally indeterminate.

For any $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$, let $\|\mathbf{x}\| := \#\{k \in \mathcal{K}; x_k = 1\}$. If all of \mathcal{K} is a single $\Gamma_{\mathcal{X}}$ -orbit (i.e. $\Gamma_{\mathcal{X}}$ acts *transitively* on \mathcal{K}), then the only $\Gamma_{\mathcal{X}}$ -fixed points in $\{0, 1\}^{\mathcal{K}}$ are $\mathbf{0} := (0, 0, \dots, 0)$ and $\mathbf{1} := (1, 1, \dots, 1)$. Thus, the conditions of Proposition 3.3 reduce to:

- (i) Either $\mathbf{0} \notin \mathcal{X}$ and there is some $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\| < |\mathcal{K}|/2$;
- (ii) or $\mathbf{1} \notin \mathcal{X}$ and there is some $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\| > |\mathcal{K}|/2$.

EXAMPLE 3.4. (*Committee Selection*) Let $0 \leq I \leq J \leq K$, and define $\mathcal{X}_{I,J;K}^{\text{com}} := \{\mathbf{x} \in \{0, 1\}^{\mathcal{K}}; I \leq \|\mathbf{x}\| \leq J\}$. Interpretationally, \mathcal{K} is a set of K ‘candidates’, and $\mathcal{X}_{I,J;K}^{\text{com}}$ is the set of all ‘committees’ comprised of at least I and at most J of these candidates. For instance, the space in Example 2.1(b) was $\mathcal{X}_{2,2;5}^{\text{com}}$.

If $\mathcal{X} = \mathcal{X}_{I,J;K}^{\text{com}}$, then it is easy to see that $\Gamma_{\mathcal{X}}$ contains all permutations of \mathcal{K} , so $\Gamma_{\mathcal{X}}$ acts transitively on \mathcal{K} . Thus, if either (i) $0 < I < K/2$ or (ii) $K/2 < J < K$, then Proposition 3.3 implies that $\mathcal{X}_{I,J;K}^{\text{com}}$ is globally indeterminate.⁹ On the other hand, if $I = 0$ and $J = 1$, or if $I = K - 1$ and $J = K$, then $\mathcal{X}_{I,J;K}^{\text{com}}$ is majority determinate. In between these extremes, if either $I = 0$ and $2 \leq J < K/2$, or $K/2 < I \leq K - 2$

⁹This fact was established in [NPP14] through a different argument.

and $J = K$, then the space $\mathcal{X}_{I,J;K}^{\text{com}}$ is majority indeterminate, but *not* globally indeterminate: although $\text{Crit}(\mathcal{X}_{I,J;K}^{\text{com}}) \neq \emptyset$, we have $\text{Maj}(\mathcal{X}_{I,J;K}^{\text{com}}) \cap \text{Crit}(\mathcal{X}_{I,J;K}^{\text{com}}) = \emptyset$, so the condition of Theorem 3.1(b) is violated. (For instance, $\text{Crit}(\mathcal{X}_{0,2;5}^{\text{com}}) = \{\mathbf{1}\}$,¹⁰ but $\mathbf{1} \notin \text{Maj}(\mathcal{X}_{0,2;5}^{\text{com}})$.) This case belongs to an intermediate category, which we might call *partially determinate*. \diamond

3.2. Global indeterminacy in McGarvey spaces. A space \mathcal{X} is called *McGarvey* if $\text{Maj}(\mathcal{X}) = \{0, 1\}^{\mathcal{K}}$ —in other words, for all $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$, there exists some $\mu \in \Delta^*(\mathcal{X})$ with $\text{Maj} = \mathbf{x}$. These spaces are studied in [NP11].¹¹ Theorem 3.1(b) has the following simple corollary.

COROLLARY 3.5. *Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ be McGarvey. Then*

$$\left(\mathcal{X} \text{ is globally indeterminate}\right) \iff \left(\text{Crit}(\mathcal{X}) \neq \emptyset\right).$$

To illustrate this result, we will consider a class of aggregation spaces representing convexity structures. For any subset $\mathcal{J} \subseteq \mathcal{K}$, we define the view $\mathbf{1}^{\mathcal{J}} \in \{0, 1\}^{\mathcal{K}}$ by setting $\mathbf{1}_j^{\mathcal{J}} := 1$ for all $j \in \mathcal{J}$ and $\mathbf{1}_k^{\mathcal{J}} := 0$ for all $k \in \mathcal{K} \setminus \mathcal{J}$. (Thus, $\mathbf{1}^{\mathcal{K}} = \mathbf{1}$.) Let \mathfrak{C} be a collection of subsets of \mathcal{K} . We say that \mathfrak{C} is a *convexity structure* if $\emptyset \in \mathfrak{C}$, $\mathcal{K} \in \mathfrak{C}$, and \mathfrak{C} is closed under intersections.¹² We then define the associated aggregation space $\mathcal{X}_{\mathfrak{C}} := \{\mathbf{1}^{\mathcal{C}}; \mathcal{C} \in \mathfrak{C}\} \subset \{0, 1\}^{\mathcal{K}}$; we will call $\mathcal{X}_{\mathfrak{C}}$ a *convexity space*.

EXAMPLE 3.6. (a) (*Linear convexity*) Let $K \in \mathbb{N}$, and suppose $\mathcal{K} := [1 \dots K]$ with the natural ordering. The *linear* convexity structure $\mathfrak{C}_K^{\text{line}}$ consists of all subintervals of \mathcal{K} . The corresponding convexity space is defined: $\mathcal{X}_K^{\text{line}} := \{\mathbf{1}^{\mathcal{C}}; \mathcal{C} \in \mathfrak{C}_K^{\text{line}}\} \subset \{0, 1\}^{\mathcal{K}}$.

(b) (*Taxonomic hierarchies*) Let \mathfrak{T} be a collection of subsets of \mathcal{K} . We say that \mathfrak{T} is a *taxonomic hierarchy* if, for all $\mathcal{C}, \mathcal{D} \in \mathfrak{T}$, we have: either (1) $\mathcal{C} \subset \mathcal{D}$, or (2) $\mathcal{D} \subset \mathcal{C}$, or (3) \mathcal{C} and \mathcal{D} are disjoint. The elements of \mathfrak{T} are called *taxa*. It is easy to see that \mathfrak{T} is a convexity structure.

Taxonomic hierarchies arise in many collective decisions. Consider, for instance, the problem of determining which group of animals should be granted animal rights. A plausible norm would be to require that this group, whatever it otherwise may be, must form a biological taxon derived from the evolutionary tree of species. The meta-standard thus allows for individual disagreement about the appropriate specific taxon, say whether it be only the mammals or all vertebrates, but no disagreement about the fact that it must be a taxon. \diamond

The following result provides a simple and frequently applicable sufficient condition for global indeterminacy in convexity spaces.¹³

PROPOSITION 3.7. *Let \mathfrak{C} be a convexity structure on \mathcal{K} . Then, $\mathcal{X}_{\mathfrak{C}}$ is McGarvey if and only if \mathfrak{C} contains all singletons. In this case, $\mathcal{X}_{\mathfrak{C}}$ is globally indeterminate.*

¹⁰See Lemma A.6.

¹¹In the setting of Arrovian preference aggregation, McGarvey [McG53] showed that any tournament could be realized as the majority tournament of some profile of preferences. Thus, in our terminology, he showed that the space $\mathcal{X}_{\mathcal{N}}^{\text{p}}$ was “McGarvey”. Stearns [Ste59] refined McGarvey’s original result. Later, Hollard and le Breton [HLB96], Vidu [Vid99], and Shelah [She09] extended it to other preference aggregation problems. Our recent paper [NP11] extends McGarvey’s result to many other judgement aggregation problems.

¹²See [vdV93] for an excellent introduction to convexity structures.

¹³We assume $\emptyset \in \mathfrak{C}$. Note that if $\emptyset \notin \mathfrak{C}$, then $\mathcal{X}_{\mathfrak{C}}$ is indeterminate in a trivial way.

EXAMPLE 3.8. (a) Let $\mathfrak{C}_K^{\text{line}}$ be the linear convexity from Example 3.6(a). Clearly, $\{k\} \in \mathfrak{C}_K^{\text{line}}$ for all $k \in \mathcal{K}$ (because $\{k\}$ is an interval). Thus, the convexity space $\mathcal{X}_K^{\text{line}}$ is globally indeterminate.

(b) Suppose \mathfrak{T} is a taxonomic hierarchy on \mathcal{K} , as in Example 3.6(b). If every singleton set is a taxon, then the corresponding convexity space $\mathcal{X}_{\mathfrak{T}}$ is globally indeterminate. \diamond

Many other McGarvey spaces are also globally indeterminate, such as the permutahedron $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$ defined at the end of Section 2. (See [NPP14] for more examples.) This suggests a strong connection between the McGarvey property and global indeterminacy. Indeed, we do not know of a single, ‘naturally occurring’ aggregation space that is McGarvey but *not* globally indeterminate. One might thus conjecture that any non-degenerate McGarvey space \mathcal{X} is globally indeterminate. But the next example falsifies this conjecture.

EXAMPLE 3.9. Let \mathcal{X} be the subset of $\{0, 1\}^5$ defined by the two critical fragments $\mathbf{w}_1 = (1, 1, 0, 0, *)$ and $\mathbf{w}_2 = (*, 0, 0, 1, 1)$.¹⁴ By construction, we have $\text{Crit}(\mathcal{X}) = \emptyset$ (since \mathbf{w}_1 and \mathbf{w}_2 disagree in the second and fourth coordinate). Moreover, \mathcal{X} has 28 elements which is more than $3/4$ of $32 = |\{0, 1\}^5|$; hence \mathcal{X} is McGarvey, by Proposition 2.4(a) of [NP11]. But by Corollary 3.5, \mathcal{X} is *not* globally indeterminate. Note also that \mathcal{X} contains $\mathbf{1}^k$ for all k ; this shows that the hypothesis of convexity (i.e. closedness under intersections) cannot be dropped in Proposition 3.7 above.

3.3. Susceptibility for indeterminacy. Global indeterminacy concerns the *existence* of a profile such that, in each issue, both answers are possible via a suitably chosen decision path. One may doubt the relevance of this concept and the corresponding analysis, since existence results might only describe ‘worst cases’ that are very special and unlikely to happen. But the earlier example of global indeterminacy in preference aggregation already suggests that global indeterminacy is far from being special and unlikely, in the sense that it is obtained for a large set of profiles. We will therefore now investigate how ‘likely’ such profiles are.

We will take the complexity of a profile as a proxy for its unlikelihood. We will assess this complexity in terms of a very simple but instructive measure: the number of voters needed to construct the profile. Formally, for any $N \in \mathbb{N}$, let

$$\Delta_N^*(\mathcal{X}) := \left\{ \mu \in \Delta^*(\mathcal{X}) ; \forall \mathbf{x} \in \mathcal{X}, \mu(\mathbf{x}) = \frac{n}{N} \text{ for some } n \in [0 \dots N] \right\}.$$

From a social choice perspective, $\Delta_N^*(\mathcal{X})$ is the set of profiles which can be generated by a population of exactly N voters. From a geometric perspective, $\Delta_N^*(\mathcal{X})$ can be visualized as a discrete ‘mesh’ of density $1/N$ embedded in the set $\Delta(\mathcal{X})$. Let $\Delta_{\text{ind}}^*(\mathcal{X}) := \{\mu \in \Delta^*(\mathcal{X}) ; \mu \text{ is globally indeterminate}\}$. We define the *susceptibility for indeterminacy* of \mathcal{X} to be

$$\eta(\mathcal{X}) := \min \{N \in \mathbb{N} ; \Delta_N^*(\mathcal{X}) \cap \Delta_{\text{ind}}^*(\mathcal{X}) \neq \emptyset\}.$$

(with $\eta(\mathcal{X}) = \infty$ if \mathcal{X} is not globally indeterminate). From a social choice perspective, $\eta(\mathcal{X})$ is the minimum number of voters needed to construct a globally indeterminate profile. From a geometric perspective, $\eta(\mathcal{X})$ places an upper bound on the ‘thickness’ of $\Delta_{\text{ind}}^*(\mathcal{X})$: if $\eta(\mathcal{X}) > N$, then $\Delta_{\text{ind}}^*(\mathcal{X})$ cannot contain a sphere

¹⁴Here, as usual, the “*” indicates an unspecified coordinate—in other words, $\text{supp}(\mathbf{w}_1) = \{1, 2, 3, 4\}$, and $\text{supp}(\mathbf{w}_2) = \{2, 3, 4, 5\}$.

of radius greater than $\frac{\epsilon}{N}$, where $\epsilon := \sqrt{1 - \frac{1}{K}}$. Thus, $\eta(\mathcal{X})$ measures the susceptibility of \mathcal{X} to global indeterminacy: if $\eta(\mathcal{X})$ is small, then \mathcal{X} is very susceptible.

Evidently, for any $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$, we have $\eta(\mathcal{X}) \geq 3$. The next four results show that several common aggregation spaces are very susceptible to global indeterminacy, since they exhibit the minimal value $\eta(\mathcal{X}) = 3$.

PROPOSITION 3.10. *If \mathcal{X} has a critical fragment of size K , then $\eta(\mathcal{X}) = 3$.*

For example, if \mathcal{X} is the truth function aggregation space from Example 2.1(d), then $\eta(\mathcal{X}) = 3$, because of the critical fragment of size 4 shown in Example 2.5(b).

Now let us consider the classical problem of Arrovian preference aggregation, as represented by judgement aggregation on the permutahedron $\mathcal{X}_{\mathcal{N}}^{\text{pf}}$, which was defined at the end of Section 2.

PROPOSITION 3.11. *If $|\mathcal{N}| \geq 3$, then $\eta(\mathcal{X}_{\mathcal{N}}^{\text{pf}}) = 3$.*

For a third example, consider the problem of aggregating equivalence relations, as formalized by [FR86]. Let \mathcal{N} be a finite set, and let \mathcal{K} be a subset of $\mathcal{N} \times \mathcal{N}$ containing exactly one of the pairs (n, m) or (m, n) for each $n \neq m \in \mathcal{N}$. Thus, an element of $\{0, 1\}^{\mathcal{K}}$ represents a symmetric, reflexive binary relation (i.e. undirected graph) on \mathcal{N} . Let $\mathcal{X}_{\mathcal{N}}^{\text{eq}} \subset \{0, 1\}^{\mathcal{K}}$ be the set of all equivalence relations on \mathcal{N} . Thus, judgement aggregation on $\mathcal{X}_{\mathcal{N}}^{\text{eq}}$ represents the problem of constructing a collective *classification* of the elements of \mathcal{N} . Each voter has in mind her own classification (represented by some equivalence relation on \mathcal{N}); our problem is to aggregate these into some collective classification.

PROPOSITION 3.12. *If $N \geq 3$, then $\eta(\mathcal{X}_{\mathcal{N}}^{\text{eq}}) = 3$.*

We now turn to the problem of resource allocation. Fix $M, D \in \mathbb{N}$, and consider the D -dimensional ‘discrete cube’ $[0 \dots M]^D$. Each element $\mathbf{x} \in [1 \dots M]^D$ can be represented by a point $\Phi(\mathbf{x}) := \tilde{\mathbf{x}} \in \{0, 1\}^{D \times M}$ defined as follows:

$$(3.2) \quad \text{for all } (d, m) \in [1 \dots D] \times [1 \dots M], \quad \tilde{x}_{(d,m)} := \begin{cases} 1 & \text{if } x_d \geq m; \\ 0 & \text{if } x_d < m. \end{cases}$$

This defines an injection $\Phi : [0 \dots M]^D \rightarrow \{0, 1\}^{D \times M}$. Any subset of $\mathcal{P} \subset [0 \dots M]^D$ can thereby be represented as a subset $\mathcal{X} := \Phi(\mathcal{P}) \subset \{0, 1\}^{D \times M}$. Judgement aggregation over \mathcal{X} thus represents social choice over a D -dimensional ‘policy space’, where each voter’s position represents her ideal point in \mathcal{P} , the set of feasible policies. This framework is especially useful for resource allocation problems, as we now illustrate. Let

$$(3.3) \quad \begin{aligned} \Delta_M^D &:= \left\{ \mathbf{x} \in [0 \dots M]^D ; \sum_{d=1}^D x_d = M \right\}, \\ \text{and } \mathcal{X}_{M,D}^\Delta &:= \Phi[\Delta_M^D] \subset \{0, 1\}^{D \times M}. \end{aligned}$$

Geometrically, Δ_M^D is a ‘discrete simplex’; points in Δ_M^D represent all ways of allocating M indivisible dollars amongst exactly D claimants. Thus, judgement aggregation over $\mathcal{X}_{M,D}^\Delta$ describes a group which decides how to allocate a budget of M dollars towards D claimants by voting ‘yes’ or ‘no’ to propositions of the form ‘ x_d should be at least m dollars’ for each $d \in [1 \dots D]$ and $m \in [1 \dots M]$; see [LNP10].

PROPOSITION 3.13. *If $D \geq 3$, then $\eta(\mathcal{X}_{M,D}^\Delta) = 3$.*

We have seen that globally indeterminate profiles are ‘easy’ to construct for $\mathcal{X}_{\mathcal{N}}^{\text{pr}}$, $\mathcal{X}_{\mathcal{N}}^{\text{eq}}$, and $\mathcal{X}_{M,D}^{\Delta}$. But the committee space $\mathcal{X}_{I,J;K}^{\text{com}}$ from Example 3.4 exhibits a more complex pattern. Consider, for instance, the space $\mathcal{X}_{4,6;10}^{\text{com}}$, i.e. the space of all committees that contain at least 4 and at most 6 members of a set of 10 candidates. Define $\mathbf{0} := (0, 0, \dots, 0)$ and $\mathbf{1} := (1, 1, \dots, 1)$. Theorem 3.1(a) says that profile $\mu \in \Delta^*(\mathcal{X}_{4,6;10}^{\text{com}})$ is globally indeterminate if and only if either $\text{Maj}(\mu) = \mathbf{0}$ or $\text{Maj}(\mu) = \mathbf{1}$ (see Lemma A.6 in the Appendix). Without loss of generality, by symmetry, suppose the former, i.e. $\tilde{\mu}_k < \frac{1}{2}$ for all $k \in [1 \dots K]$. Since each feasible view endorses at least 4 candidates, we have $\sum_k \tilde{\mu}_k \geq 4$. Denoting by k^* the candidate with maximal popular support, we thus obtain $\frac{4}{10} \leq \tilde{\mu}_{k^*} < \frac{1}{2}$. Satisfaction of this inequality requires at least five agents; together with Proposition 3.14(b) below we thus obtain $\eta(\mathcal{X}_{4,6;10}^{\text{com}}) = 5$.

The argument just given can be generalized to give the lower bound on $\eta(\mathcal{X}_{I,J;K}^{\text{com}})$ in part (a) of the following result.

PROPOSITION 3.14. (a) *If $0 < I \leq J < K$, then*

$$(3.4) \quad \eta(\mathcal{X}_{I,J;K}^{\text{com}}) \geq \min \left\{ \frac{K}{K-2I}, \frac{K}{2J-K} \right\}.$$

(b) *For any $0 \leq I \leq J \leq K$, if $I < K/2$ or $J > K/2$, then $\eta(\mathcal{X}_{I,J;K}^{\text{com}}) \leq K$. Moreover, one obtains the following upper bounds for $\eta(\mathcal{X}_{I,J;K}^{\text{com}})$.*

- [i] *If $0 < I < K/2$, then let $N := \left\lceil \frac{I}{K-2I} \right\rceil$. Then $\eta(\mathcal{X}_{I,J;K}^{\text{com}}) \leq 2N + 1$.*
- [ii] *If $K/2 < J$, then let $N := \left\lceil \frac{K-J}{2J-K} \right\rceil$. Then $\eta(\mathcal{X}_{I,J;K}^{\text{com}}) \leq 2N + 1$.*

Combining parts (a) and (b) of Proposition 3.14, we obtain $\eta(\mathcal{X}_{4,6;10}^{\text{com}}) = 5$, as noted above; similarly, one obtains, for example, $\eta(\mathcal{X}_{6,8;14}^{\text{com}}) = 7$ and $\eta(\mathcal{X}_{5,6;11}^{\text{com}}) = 11$. Note that the hypothesis that $I < K/2$ or $J > K/2$ cannot be eliminated from part (b). If $I = K/2 = J$, then $\mathcal{X}_{I,J;K}^{\text{com}}$ is *not* globally indeterminate (so $\eta(\mathcal{X}_{I,J;K}^{\text{com}}) = \infty$ in this case).

4. Full indeterminacy

Global indeterminacy means that, for some profile, any answer can be obtained on each issue by choosing a suitable decision path. An even stronger form of indeterminacy, which we shall henceforth refer to as *full* indeterminacy, occurs if, for some profile, any logically possible *combination* of answers across issues can be obtained via an appropriate decision path, i.e. if the corresponding Condorcet set contains *all* possible views.

Formally, a profile $\mu \in \Delta^*(\mathcal{X})$ is *fully indeterminate* if $\text{Cond}(\mathcal{X}, \mu) = \mathcal{X}$. We say that \mathcal{X} is *fully indeterminate* if there exists some $\mu \in \Delta^*(\mathcal{X})$ which is fully indeterminate.¹⁵

¹⁵For all $k \in \mathcal{K}$, suppose that there is some $\mathbf{x} \in \mathcal{X}$ with $x_k = 1$, and some $\mathbf{x}' \in \mathcal{X}$ with $x'_k = 0$. Then full indeterminacy of \mathcal{X} implies global indeterminacy. Thus, under a very mild and natural hypothesis (which is almost always satisfied in practice), full indeterminacy is a logically stronger property than global indeterminacy. However, we do not actually need this hypothesis for any of the results in this paper, so we will not assume it in what follows.

EXAMPLE 4.1. Fix $J \in (\frac{K}{2} \dots K]$, and let $\mathcal{X}_{J,J;K}^{\text{com}} := \{\mathbf{x} \in \{0,1\}^{\mathcal{K}}; \|\mathbf{x}\| = J\}$. (Thus, $\mathcal{X}_{J,J;K}^{\text{com}}$ is the set of all ‘committees’ comprised of exactly J out of K candidates.) Let μ be the uniform distribution on $\mathcal{X}_{J,J;K}^{\text{com}}$. Then $\text{Cond}(\mathcal{X}_{J,J;K}^{\text{com}}, \mu) = \mathcal{X}_{J,J;K}^{\text{com}}$, hence μ is fully indeterminate. (The proof is straightforward: if there are exactly J open slots and K viable candidates, and the slots are allocated on a ‘first come, first serve’ basis, then the slots will simply be allocated to the first J candidates.) \diamond

Now let $D \geq 3$, let $M \in \mathbb{N}$, and let $\mathcal{X}_{M,D}^{\Delta}$ be as in eqn.(3.3) of Section 3. We define $\mathbf{0} := (0, 0, \dots, 0) \in \{0,1\}^{\mathcal{K}}$.

PROPOSITION 4.2. *Let $\mu \in \Delta^*(\mathcal{X}_{M,D}^{\Delta})$. Then μ is fully indeterminate if and only if $\text{Maj}(\mu) = \mathbf{0}$.*

For example, for all $d \in [1 \dots D]$, let \mathbf{x}^d be the element of $\mathcal{X}_{M,D}^{\Delta}$ which allocates all M dollars towards claimant d . (Thus, for all $m \in [1 \dots M]$, we have $x_{d,m}^d = 1$, while $x_{c,m}^d = 0$ for all $c \in [1 \dots D] \setminus \{d\}$.) Let $\mu \in \Delta^*(\mathcal{X}_{M,D}^{\Delta})$ be the profile which allocates weight $1/D$ to each of $\mathbf{x}^1, \dots, \mathbf{x}^D$. Then $\text{Maj}(\mu) = \mathbf{0}$, so Proposition 4.2 implies that μ is fully indeterminate.¹⁶

For any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{z} \in \{0,1\}^{\mathcal{K}}$, recall that $\mathbf{x} \succ \mathbf{z}$ if there is no $\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}$ which is *between* \mathbf{x} and \mathbf{z} (i.e. such that, for all $k \in \mathcal{K}$ and $c \in \{0,1\}$, $(x_k = c = z_k) \implies (y_k = c)$). We say that $\mathbf{z} \in \{0,1\}^{\mathcal{K}}$ is a *panopticon* for \mathcal{X} if $\mathbf{x} \succ \mathbf{z}$ for all $\mathbf{x} \in \mathcal{X}$ (this implies that $\mathbf{z} \notin \mathcal{X}$, because any element of \mathcal{X} is between itself and every other element of \mathcal{X}). Heuristically, from a panopticon, one can ‘see’ each element of \mathcal{X} without the view being blocked by any other elements. For example, $\mathbf{1}$ is a panopticon for $\mathcal{X}_{J,J;K}^{\text{com}}$. Let $\text{Pan}(\mathcal{X})$ be the set of all panoptica for \mathcal{X} . The next result is this section’s key observation:

PROPOSITION 4.3. *Let $\mathcal{X} \subseteq \{0,1\}^{\mathcal{K}}$.*

- (a) *For any $\mu \in \Delta^*(\mathcal{X})$, $(\mu \text{ is fully indeterminate}) \iff (\text{Maj}(\mu) \in \text{Pan}(\mathcal{X}))$.*
- (b) $(\mathcal{X} \text{ is fully indeterminate}) \iff (\text{Maj}(\mathcal{X}) \cap \text{Pan}(\mathcal{X}) \neq \emptyset)$.

For any $\mathbf{x}, \mathbf{y} \in \{0,1\}^{\mathcal{K}}$, we define their *Hamming distance* by $d_H(\mathbf{x}, \mathbf{y}) := \#\{k \in \mathcal{K}; x_k \neq y_k\}$. Note that $\text{Pan}(\mathcal{X}) \neq \emptyset$ requires that $d_H(\mathbf{x}, \mathbf{y}) \geq 2$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. This is in itself a strong restriction; for instance, it precludes the permutahedron from having a panopticon, and hence from being fully indeterminate. Above, we have seen that the ‘allocation’ problems on $\mathcal{X}_{J,J;K}^{\text{com}}$ and $\mathcal{X}_{M,D}^{\Delta}$ are fully indeterminate. Are there other natural classes of aggregation spaces that are fully indeterminate? A necessary condition is that $\text{Pan}(\mathcal{X}) \neq \emptyset$. However, this is not sufficient, as shown by the next counterexample.

EXAMPLE 4.4. Let $M \geq 3$ and let $\mathcal{X}_{M,2}^{\Delta}$ be as defined in eqn.(3.3) of §3. Then $\text{Pan}(\mathcal{X}_{M,2}^{\Delta}) = \{\mathbf{0}, \mathbf{1}\}$. (Intuitively: $\mathbf{0}$ and $\mathbf{1}$ correspond to the unfeasible allocations $(0, 0)$ and (M, M) , respectively.) However, $\mathcal{X}_{M,2}^{\Delta}$ is a median space,

¹⁶This profile is not as contrived as it may seem. When the government engages in wealth redistribution, the elements of $[1 \dots G]$ represent the potential recipients of government largesse (e.g. state governments seeking federal assistance; economic sectors seeking subsidies, etc.). If the redistribution is decided by a committee (e.g. the Senate), and each potential recipient controls roughly the same number of committee members (e.g. each state has two senators), then the resulting profile closely resembles this fully indeterminate profile.

so $\text{Maj}(\mathcal{X}_{M,2}^\Delta) = \mathcal{X}_{M,2}^\Delta$. (The argument is very similar to Example 2.5(a).) Thus, $\mathcal{X}_{M,2}^\Delta$ is *not* fully indeterminate. \diamond

Recall from §3.2 that a space \mathcal{X} is called *McGarvey* if $\text{Maj}(\mathcal{X}) = \{0, 1\}^\mathcal{K}$. If \mathcal{X} is McGarvey, then \mathcal{X} is fully indeterminate if and only if $\text{Pan}(\mathcal{X}) \neq \emptyset$. However, it is not clear that the McGarvey property is compatible with $\text{Pan}(\mathcal{X}) \neq \emptyset$. Heuristically, the problem is that, to have $\text{Pan}(\mathcal{X}) \neq \emptyset$, the space \mathcal{X} must be a relatively ‘small’ subset of $\{0, 1\}^\mathcal{K}$, whereas to be McGarvey, \mathcal{X} must be relatively ‘large’. The next proposition illustrates this conflict. Define $\mathbf{1} := (1, 1, \dots, 1) \in \{0, 1\}^\mathcal{K}$. For any $k \in \mathcal{K}$, recall the definition of the view $\mathbf{1}^k \in \{0, 1\}^\mathcal{K}$ from §3.2.

PROPOSITION 4.5. *Suppose \mathcal{X} contains $\mathbf{1}$ and $\mathbf{1}^k$, for all $k \in \mathcal{K}$. Then \mathcal{X} is McGarvey, but $\text{Pan}(\mathcal{X}) = \emptyset$ (so \mathcal{X} is not fully indeterminate).*

EXAMPLE 4.6. Recall the space $\mathcal{X}_\mathcal{N}^{\text{eq}}$ from §3.3. Note that $\mathbf{1} \in \mathcal{X}_\mathcal{N}^{\text{eq}}$ (representing the relation where all elements of \mathcal{N} are equivalent). Also, for all $n, m \in \mathcal{N}$, we have $\mathbf{1}^{(n,m)} \in \mathcal{X}_\mathcal{N}^{\text{eq}}$ (representing the equivalence relation where $n \sim m$ and all other elements are non-equivalent). Thus, Proposition 4.5 implies that $\mathcal{X}_\mathcal{N}^{\text{eq}}$ is McGarvey, but not fully indeterminate. \diamond

Propositions 4.3(b) and 4.5 together suggest that many naturally occurring aggregation spaces will *not* be fully indeterminate (although they may still be globally indeterminate via Theorem 3.1). On the other hand, we will see in §6 that aggregation spaces will frequently be ‘almost’ fully indeterminate.

Full indeterminacy also arises as a consequence of symmetry in the judgement aggregation problem. Let $\mathcal{X} \subset \{0, 1\}^\mathcal{K}$ and let $\Gamma_\mathcal{X}$ be the group of all symmetries of \mathcal{X} , as defined in §3.1. For any $\mathbf{x} \in \{0, 1\}^\mathcal{K}$, the $\Gamma_\mathcal{X}$ -orbit of \mathbf{x} is the set $\Gamma(\mathbf{x}) := \{\gamma_*(\mathbf{x}); \gamma \in \Gamma_\mathcal{X}\}$. Distinct $\Gamma_\mathcal{X}$ -orbits are disjoint. Thus, \mathcal{X} is a disjoint union of $\Gamma_\mathcal{X}$ -orbits. We say \mathcal{X} is *homogeneous* if all of \mathcal{X} is contained in one $\Gamma_\mathcal{X}$ -orbit. (For example, $\mathcal{X}_{J,J;K}^{\text{com}}$ is homogeneous; however, $\mathcal{X}_\mathcal{N}^{\text{pr}}$ is not.)

For any $k \in \mathcal{K}$, recall that the $\Gamma_\mathcal{X}$ -orbit of k is the set $\{\gamma(k); \gamma \in \Gamma_\mathcal{X}\}$, and \mathcal{K} is a disjoint union of $\Gamma_\mathcal{X}$ orbits — call them $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_N$. For any $\mathbf{x} \in \mathcal{X}$ and any $n \in [1 \dots N]$, let $\|\mathbf{x}_{\mathcal{K}_n}\| := \#\{k \in \mathcal{K}_n; x_k = 1\}$. It is easy to see that $\|\mathbf{x}_{\mathcal{K}_n}\| = \|\gamma(\mathbf{x})_{\mathcal{K}_n}\|$ for any $\gamma \in \Gamma_\mathcal{X}$. Thus, if \mathcal{X} is homogeneous, then $\|\mathbf{x}_{\mathcal{K}_n}\| = \|\mathbf{y}_{\mathcal{K}_n}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

PROPOSITION 4.7. *Suppose \mathcal{X} is homogeneous, and for all $n \in [1 \dots N]$, we have $\|\mathbf{x}_{\mathcal{K}_n}\| \neq |\mathcal{K}_n|/2$ for some (and hence, all) $\mathbf{x} \in \mathcal{X}$. Then \mathcal{X} is fully indeterminate.*

EXAMPLE 4.8. Let \mathcal{L} and \mathcal{R} be two finite sets, and let $\mathcal{K} := \mathcal{L} \times \mathcal{R}$. Thus, an element of $\{0, 1\}^\mathcal{K}$ can be interpreted as a bipartite graph, where the vertices are partitioned into the sets \mathcal{L} and \mathcal{R} . Suppose $R := |\mathcal{R}|$ is odd. For all $\ell \in \mathcal{L}$, let $R_\ell \in [0 \dots R]$, and let $\mathcal{K}_\ell := \{\ell\} \times \mathcal{R} \subset \mathcal{K}$. Finally, for any $\mathbf{x} \in \mathcal{X}$, define $\|\mathbf{x}_{\mathcal{K}_\ell}\| := \#\{k \in \mathcal{K}_\ell; x_k = 1\}$ (i.e. the number of edges from the vertex ℓ into the set \mathcal{R} in the graph represented by \mathbf{x}). Define

$$\mathcal{X} := \{\mathbf{x} \in \{0, 1\}^\mathcal{K}; \|\mathbf{x}_{\mathcal{K}_\ell}\| = R_\ell \text{ for all } \ell \in \mathcal{L}\}.$$

Any permutation of \mathcal{R} induces a permutation of \mathcal{K} in the obvious way, and each of these is a symmetry of \mathcal{X} . If the elements $\{R_\ell\}_{\ell \in \mathcal{L}}$ are all distinct, then these permutations are the only symmetries of \mathcal{X} . For all $\ell \in \mathcal{L}$, we have $|\mathcal{K}_\ell| = R$, while $\|\mathbf{x}_{\mathcal{K}_\ell}\| \neq R/2$ for all $\mathbf{x} \in \mathcal{X}$ (because R is odd). Thus, Proposition 4.7 says that \mathcal{X} is fully indeterminate. \diamond

We say that $\Gamma_{\mathcal{X}}$ acts *transitively* on \mathcal{K} if all of \mathcal{K} is a single $\Gamma_{\mathcal{X}}$ -orbit.¹⁷ For the next two results, let $K := |\mathcal{K}|$.

PROPOSITION 4.9. *Suppose $\Gamma_{\mathcal{X}}$ is transitive, and there is some constant $C \neq K/2$ such that $\|\mathbf{x}\| = C$ for all $\mathbf{x} \in \mathcal{X}$. Then \mathcal{X} is fully indeterminate.*

EXAMPLE 4.10. (Trees) Let \mathcal{N} be a finite set, and let $\mathcal{K} \subset \mathcal{N} \times \mathcal{N}$ be as in the definition of $\mathcal{X}_{\mathcal{N}}^{\text{eq}}$ in §3.3. Thus, an element of $\{0, 1\}^{\mathcal{K}}$ represents a graph on the vertex set \mathcal{N} . Any permutation γ of \mathcal{N} induces a permutation γ_* of \mathcal{K} (where $\gamma_*(n, m) := (\gamma(n), \gamma(m))$). Let Γ_* denote this set of permutations of \mathcal{K} . Then Γ_* acts transitively on \mathcal{K} .

Let \mathcal{X} be the set of all elements of $\{0, 1\}^{\mathcal{K}}$ representing *trees* on the vertex set \mathcal{N} —that is, connected graphs which become disconnected after the removal of any single edge. If $N := |\mathcal{N}|$, then every tree on \mathcal{N} has exactly $N - 1$ edges. Thus, $\|\mathbf{x}\| = N - 1$ for all $\mathbf{x} \in \mathcal{X}$. Note that $N - 1 \neq K/2$ (because $K = N(N - 1)/2$). Finally, it is clear that $\Gamma_* \subseteq \Gamma_{\mathcal{X}}$. Thus, $\Gamma_{\mathcal{X}}$ acts transitively on \mathcal{K} . Thus, Proposition 4.9 says that \mathcal{X} is fully indeterminate. (Note that this space is *not* homogeneous, so Proposition 4.7 is not applicable.) \diamond

The next result can be seen as a consequence of either Proposition 4.7 or Proposition 4.9.

COROLLARY 4.11. *Suppose \mathcal{X} is homogeneous, and $\Gamma_{\mathcal{X}}$ acts transitively on \mathcal{K} , and there is some $\mathbf{x} \in \mathcal{X}$ with $\|\mathbf{x}\| \neq K/2$. Then \mathcal{X} is fully indeterminate.*

EXAMPLE 4.12. Let \mathcal{N} be a finite set, and let $\mathcal{K} \subset \mathcal{N} \times \mathcal{N}$ be as in Example 4.10, so that an element of $\{0, 1\}^{\mathcal{K}}$ represents a graph on the vertex set \mathcal{N} .

(a) (*Cycles*) Let $\mathcal{X} \subset \{0, 1\}^{\mathcal{K}}$ represent the set of all *cycles*—that is, connected graphs with exactly $|\mathcal{N}|$ edges which remain connected when any single edge is removed. Then $\Gamma_* \subseteq \Gamma_{\mathcal{X}}$ and acts transitively on \mathcal{X} , so \mathcal{X} is homogeneous. Thus, Corollary 4.11 says \mathcal{X} is fully indeterminate. A similar argument holds if \mathcal{X} is the space of all *lines* (i.e. connected graphs obtained by removing one edge from a cycle).

(b) (*Multicliques*) Let $J \geq 2$ and $N_1, \dots, N_J \geq 1$ be integers such that $N_1 + \dots + N_J = |\mathcal{N}|$, and let \mathcal{X} be the space of all graphs made of a disjoint union of J *cliques* (i.e. complete subgraphs) with cardinalities N_1, \dots, N_J , with no edges between them. Once again, $\Gamma_* \subseteq \Gamma_{\mathcal{X}}$ and acts transitively on \mathcal{X} , so \mathcal{X} is homogeneous. Thus, Corollary 4.11 says \mathcal{X} is fully indeterminate. \diamond

5. Generalized Antichains

By Proposition 4.3, a necessary condition for full indeterminacy is that the set of panoptica is non-empty. It is therefore desirable to find structural conditions on \mathcal{X} which guarantee that $\text{Pan}(\mathcal{X}) \neq \emptyset$. For any $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, define $\mathbf{x} \oplus \mathbf{y} := \mathbf{z} \in \{0, 1\}^{\mathcal{K}}$ by $z_k := (x_k + y_k) \bmod 2$, for all $k \in \mathcal{K}$. For any $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$, define the involution $I_{\mathbf{x}} : \{0, 1\}^{\mathcal{K}} \rightarrow \{0, 1\}^{\mathcal{K}}$ by

$$(5.1) \quad I_{\mathbf{x}}(\mathbf{y}) \quad := \quad \mathbf{x} \oplus \mathbf{y}, \quad \text{for all } \mathbf{y} \in \{0, 1\}^{\mathcal{K}}.$$

¹⁷Note that such transitivity is neither necessary nor sufficient for \mathcal{X} to be homogeneous. In Example 4.8, \mathcal{X} is homogeneous but $\Gamma_{\mathcal{X}}$ does not act transitively on \mathcal{K} . In Example 4.10, $\Gamma_{\mathcal{X}}$ acts transitively on \mathcal{K} , but \mathcal{X} is not homogeneous.

Thus, $I_{\mathbf{x}}$ simply acts on $\{0, 1\}^{\mathcal{K}}$ by ‘inverting’ certain coordinates and leaving the rest alone. For any $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, write $\mathbf{x} \leq \mathbf{y}$ if $x_k \leq y_k$ for all $k \in \mathcal{K}$. A subset $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ is an *antichain* if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have $\mathbf{x} \not\leq \mathbf{y}$. (For example, $\mathcal{X}_{J, J; K}^{\text{com}}$ is an antichain.) We say \mathcal{X} is a *generalized antichain* if $I_{\mathbf{z}}[\mathcal{X}]$ is an antichain for some $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$.

PROPOSITION 5.1. $\text{Pan}(\mathcal{X}) \neq \emptyset$ if and only if \mathcal{X} is a generalized antichain.

PROPOSITION 5.2. (a) If $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ is fully indeterminate, then \mathcal{X} is a generalized antichain.

(b) Let K be odd, and let $J := \lfloor K/2 \rfloor$. Then the largest fully indeterminate subsets of $\{0, 1\}^{\mathcal{K}}$ have cardinality $\binom{K}{J}$.

Note that the converse to Corollary 5.2(a) is false: not every generalized antichain is fully indeterminate, as Example 4.4 shows. Another counterexample involves aggregation spaces representing *taxonomic hierarchies*, the convexity spaces defined in Example 3.6(b). Given a taxonomic hierarchy \mathfrak{T} on the set \mathcal{K} , we define $\mathcal{X}_{\mathfrak{T}} := \{\mathbf{1}^{\mathcal{T}}; \mathcal{T} \in \mathfrak{T}\} \subseteq \{0, 1\}^{\mathcal{K}}$. A taxon $\mathcal{T} \in \mathfrak{T}$ is *minimal* if \mathcal{T} does not contain any proper sub-taxa. Note that minimal taxa are not necessarily singletons.

PROPOSITION 5.3. Let \mathfrak{T} be a taxonomic hierarchy on \mathcal{K} .

(a) $\mathcal{X}_{\mathfrak{T}}$ is a generalized antichain if and only if every non-minimal taxon in \mathfrak{T} contains at least two non-singleton minimal taxa.

(b) However, $\mathcal{X}_{\mathfrak{T}}$ is never fully indeterminate.

We conclude this section with some necessary and sufficient conditions for \mathcal{X} to be a generalized antichain. Let $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$. Recall that $d_H(\mathbf{x}, \mathbf{y}) := \#\{k \in \mathcal{K}; x_k \neq y_k\}$ is the *Hamming distance* from \mathbf{x} to \mathbf{y} . We say that \mathbf{x} and \mathbf{y} are *adjacent* if $d_H(\mathbf{x}, \mathbf{y}) = 1$. The next result says that any sufficiently ‘dispersed’ subset of $\{0, 1\}^{\mathcal{K}}$ will be a generalized antichain.

PROPOSITION 5.4. Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$.

(a) Suppose \mathcal{X} is a generalized antichain. If $\mathbf{x} \in \mathcal{X}$, and $\mathbf{y} \in \{0, 1\}^{\mathcal{K}}$ is adjacent to \mathbf{x} , then $\mathbf{y} \notin \mathcal{X}$.

(b) If $\sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{X}} 2^{-d_H(\mathbf{x}, \mathbf{y})} < 1$, then \mathcal{X} is a generalized antichain.

As an example application of Proposition 5.4(b), consider a *random subset* of $\{0, 1\}^{\mathcal{K}}$ —that is, a collection $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$, where $\mathbf{x}_1, \dots, \mathbf{x}_N$ are independent, uniformly-distributed, $\{0, 1\}^{\mathcal{K}}$ -valued random variables. The next result says that ‘small’ random subsets will probably be generalized antichains.

PROPOSITION 5.5. For all $K \in \mathbb{N}$, let P_K be the probability that a random subset $\mathcal{X} \subset \{0, 1\}^{\mathcal{K}}$ is a generalized antichain, conditional on $|\mathcal{X}| < (\frac{4}{3})^{K/2}$. Then $\lim_{K \rightarrow \infty} P_K = 1$.

Any subset of a generalized antichain is also a generalized antichain. Also, a Cartesian product of two or more generalized antichains will be a generalized antichain (these statements are easily verified). In particular, any subset \mathcal{X} of $\mathcal{X}_{N_1, N_1; K_1}^{\text{com}} \times \mathcal{X}_{N_2, N_2; K_2}^{\text{com}} \times \dots \times \mathcal{X}_{N_J, N_J; K_J}^{\text{com}}$ (for any $N_1 < K_1, N_2 < K_2, \dots, N_J < K_J \in \mathbb{N}$) will be a generalized antichain. (Such a subset represents a ‘committee selection’ problem requiring exact representation levels from various subpopulations,

and perhaps additional constraints.) If \mathcal{X} is “large enough” (e.g. satisfies the hypotheses of Propositions 4.7 or 4.9), then $\text{Maj}(\mathcal{X}) \cap \text{Pan}(\mathcal{X}) \neq \emptyset$, so that \mathcal{X} is fully indeterminate.

6. Condorcet entropy and almost full indeterminacy

In this section, we propose a way to measure the relative size of the Condorcet set. For any subset $\mathcal{Y} \subseteq \{0, 1\}^{\mathcal{K}}$, the value $\log_2 |\mathcal{Y}|$ can be interpreted as the “information content” of \mathcal{Y} ; it is sometimes called the *entropy* of \mathcal{Y} [CT91]. Thus, for any profile $\mu \in \Delta^*(\mathcal{X})$, we define the *Condorcet entropy* of μ as

$$h(\mu) := \frac{\log_2 |\text{Cond}(\mathcal{X}, \mu)|}{\log_2 |\mathcal{X}|}.$$

That is, $h(\mu)$ measures the information content of the subset $\text{Cond}(\mathcal{X}, \mu)$, relative to the “potential” information content of all of \mathcal{X} . If $h(\mu) \approx 1$, then the profile μ is “almost” fully indeterminate, in the sense that the information content of $\text{Cond}(\mathcal{X}, \mu)$ exhausts almost all of the potential information in \mathcal{X} .

For any $\mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, let $\mathcal{X}(\mathbf{y}) := \{\mathbf{x} \in \mathcal{X} ; \mathbf{x} \succ \mathbf{y}\}$. If $\mathbf{y} = \text{Maj}(\mu)$, then Lemma 2.3(b) says that $\mathcal{X}(\mathbf{y}) = \text{Cond}(\mathcal{X}, \mu)$. Recall that $\text{Maj}[\mathcal{X}] := \bigcup\{\text{Maj}(\mu) ; \mu \in \Delta^*(\mathcal{X})\}$. We then define the *Condorcet entropy* of \mathcal{X} :

$$h(\mathcal{X}) := \sup_{\mu \in \Delta^*(\mathcal{X})} h(\mu) = \max_{\mathbf{y} \in \text{Maj}[\mathcal{X}]} \frac{\log_2 |\mathcal{X}(\mathbf{y})|}{\log_2 |\mathcal{X}|}.$$

The Condorcet entropy thus measures how close \mathcal{X} is to being fully indeterminate. In particular, $h(\mathcal{X}) = 1$ if and only if \mathcal{X} is *fully* indeterminate. Note that $h(\mathcal{X})$ reflects the combinatorial structure of \mathcal{X} , not its size. For instance, if \mathcal{X}_1 and \mathcal{X}_2 are two aggregation spaces, and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ is their Cartesian product, then $h(\mathcal{X})$ is a weighted average (*not* the sum) of $h(\mathcal{X}_1)$ and $h(\mathcal{X}_2)$.

Many of the aggregation spaces we have considered can be arranged into infinite sequences, indexed by some size parameter. For example, we can consider the sequence $\{\mathcal{X}_N^{\text{pr}}\}_{N=1}^{\infty}$, where for each $N \in \mathbb{N}$, $\mathcal{X}_N^{\text{pr}}$ is the space of all preference relations on N alternatives, as defined at the end of §2. Or we can consider the sequence $\{\mathcal{X}_N^{\text{eq}}\}_{N=1}^{\infty}$, where for each $N \in \mathbb{N}$, $\mathcal{X}_N^{\text{eq}}$ is the space of all equivalence relations on a set of N elements, as defined in §3.3. We will say that such a sequence $\{\mathcal{X}_n\}_{n=1}^{\infty}$ of aggregation problems is *asymptotically fully indeterminate* if $\lim_{n \rightarrow \infty} h(\mathcal{X}_n) = 1$.

PROPOSITION 6.1. **(a)** *The sequence $\{\mathcal{X}_N^{\text{pr}}\}_{N=1}^{\infty}$ is asymptotically fully indeterminate.*

(b) *The sequence $\{\mathcal{X}_N^{\text{eq}}\}_{N=1}^{\infty}$ is asymptotically fully indeterminate.*

Our final two examples of asymptotic full indeterminacy involve sequences of convexity spaces (see §3.2). One of these examples is the sequence of *linear* convexity spaces $\{\mathcal{X}_K^{\text{line}}\}_{K=1}^{\infty}$ defined in Example 3.6(a). To define the other example, let $D \in \mathbb{N}$, let $K = 2^D$, and identify \mathcal{K} with the hypercube $\{0, 1\}^D$ in some way. The *hypercube convexity* \mathfrak{C}_D^{\square} on $\{0, 1\}^D$ consists of all *subcubes* of $\{0, 1\}^D$. Formally, a set $C \subseteq \{0, 1\}^D$ belongs to \mathfrak{C}_D^{\square} if and only if $C = \{0, 1\}^{D_1} \times \{0\}^{D_2} \times \{1\}^{D_3}$ for some partition $\{D_1, D_2, D_3\}$ of D . Let $\mathcal{X}_D^{\square} := \{\mathbf{1}^C ; C \in \mathfrak{C}_D^{\square}\}$ (so that $\mathcal{X}_D^{\square} \subseteq \{0, 1\}^{\mathcal{K}}$ with $K = 2^D$).

- PROPOSITION 6.2. **(a)** *The sequence $\{\mathcal{X}_K^{\text{line}}\}_{K=1}^\infty$ is asymptotically fully indeterminate.*
- (b)** *The sequence $\{\mathcal{X}_D^\square\}_{D=1}^\infty$ is asymptotically fully indeterminate.*

REMARK. In Propositions 6.1(a,b) and 6.2(a), we only establish $1 - h(\mathcal{X}_N) \leq \mathcal{O}\left(\frac{1}{\log_2(N)}\right)$, so the convergence may be slow. By contrast, the convergence in 6.2(b) is in fact very rapid with $1 - h(\mathcal{X}_D^\square) \leq \mathcal{O}\left(\frac{1}{D}\right)$.

Conclusion

The Condorcet set includes all minimally acceptable compromises between majoritarianism and logical consistency. It also provides a compact description of the possible outcomes of sequential majority rule: the way in which real social decisions often emerge from an uncoordinated sequence of *ad hoc* judgements unfolding over time. Unfortunately, the Condorcet set is quite large for almost any nontrivial judgement aggregation problem. In many plausible scenarios, path-dependence can make the truth value of *every* proposition susceptible to manipulation. In some cases, *any* logically consistent outcome can arise from a suitably chosen path. In short: history matters.

Several problems remain open. For example, let \mathcal{P} be the set of all paths on \mathcal{K} . For any $\mu \in \Delta^*(\mathcal{X})$, sequential majority rule defines a function $F : \mathcal{P} \rightarrow \mathcal{X}$. Let ν be the uniform probability distribution on \mathcal{P} ; what is the distribution of $F(\nu)$? If $F(\nu)$ is almost-uniformly distributed on \mathcal{X} , this represents an especially acute form of full indeterminacy. On the other hand, if $F(\nu)$ is mostly concentrated on one or a few views, then this perhaps recommends these views as superior social choices.

We have focused on sequential *majority* vote, because the majority view has several significant properties [May52, DL10]. However, we could obtain greater path-independence by allowing some propositions to remain undecided (e.g. by using supermajoritarian voting in some coordinates), sacrificing anonymity (e.g. by using weighted voting rules) or both (e.g. by using a system with vetoes or oligarchies). In particular, if we use a system of voting rules satisfying the *intersection property*, then the outcome is guaranteed to be logically consistent, and hence, path-independent [NP07, Proposition 3.4]. For example, for all $k \in \mathcal{K}$, let N_k be the size of the largest critical fragment containing k , and let $q_k := \max\{\frac{1}{2}, 1 - \frac{1}{N_k}\}$. Suppose we decide the truth value of k via q_k -supermajoritarian voting for each $k \in \mathcal{K}$; then the outcome will be path-independent [NP07, Fact 3.4].

Is there an optimal tradeoff between decisiveness, neutrality, anonymity, and path-independence? One possibility: simple majorities could make ‘provisional’ rulings on the truth of certain propositions, but these rulings would only be treated as ‘precedents’ (i.e. binding on later decisions) if they exceeded some supermajority threshold — otherwise they could be overturned by a later, larger supermajority.

Appendix: Proofs

Proofs from Section 3.

Proof of Proposition 3.3. Let $\mathbf{x} \in \mathcal{X}$ be the element described in hypothesis (b) of Proposition 3.3. Let $\delta_{\mathbf{x}} \in \Delta^*(\mathcal{X})$ be the profile which assigns mass 1 to \mathbf{x} , and

let

$$\mu := \frac{1}{|\Gamma_{\mathcal{X}}|} \sum_{\gamma \in \Gamma_{\mathcal{X}}} \delta_{\gamma(\mathbf{x})}.$$

It is easy to verify that $\text{Maj}(\mu) = \mathbf{z}$.

Claim 1: $\mathbf{z} \in \text{Crit}(\mathcal{X})$.

Proof: Let $n \in [1 \dots N]$. The fragment $\mathbf{z}_{\mathcal{K}_n}$ is forbidden, by hypothesis (a).

So it contains a critical fragment, say \mathbf{w}_n . We must have $|\mathbf{w}_n| \geq 3$ (because $\text{Maj}(\mu) = \mathbf{z}$, but a critical fragment of size two cannot receive majority support in both coordinates). For all $\gamma \in \Gamma_{\mathcal{X}}$, the fragment $\gamma(\mathbf{w}_n)$ is also critical (because $\gamma(\mathcal{X}) = \mathcal{X}$), and is also a fragment of \mathbf{z} (since $\gamma(\mathbf{z}) = \mathbf{z}$). But \mathcal{K}_n is a $\Gamma_{\mathcal{X}}$ -orbit, so the family $\{\gamma(\mathbf{w}_n); \gamma \in \Gamma_{\mathcal{X}}\}$ covers all of $\mathbf{z}_{\mathcal{K}_n}$.

This argument works for each $n \in [1 \dots N]$. We conclude that \mathbf{z} is covered by critical fragments, as desired. \diamond claim 1

At this point, Theorem 3.1(a) implies that μ (and thus, \mathcal{X}) is fully indeterminate. \square

The proof of Proposition 3.7 requires some background about convexity structures. Let \mathfrak{C} be a convexity structure on \mathcal{K} . For any subset $\mathcal{J} \subseteq \mathcal{K}$, we define its *convex hull*:

$$\text{conv}(\mathcal{J}) := \bigcap \{\mathcal{C} \in \mathfrak{C}; \mathcal{J} \subseteq \mathcal{C}\}.$$

Note that $\mathcal{I} \subseteq \mathcal{J}$ implies $\text{conv}(\mathcal{I}) \subseteq \text{conv}(\mathcal{J})$. Thus,

$$(A1) \quad \bigcup_{\mathcal{I} \subsetneq \mathcal{J}} \text{conv}(\mathcal{I}) \subseteq \text{conv}(\mathcal{J}).$$

Say that \mathcal{J} is *Carathéodory-independent* if the set inclusion (A1) is strict [vdV93, Chapter II, §1.1, p.161]. The proof of Proposition 3.7 requires the following observation:

LEMMA A.1. *Let \mathfrak{C} be a convexity structure on \mathcal{K} . Any $\mathcal{X}_{\mathfrak{C}}$ -critical fragment has the form $\mathbf{w} = (\mathbf{1}_{\mathcal{J}}^*, 0_k)$, where $\mathcal{J} \subset \mathcal{K}$ is a Carathéodory-independent set and $k \in \mathcal{K} \setminus \text{conv}(\mathcal{J})$.*

Proof: Let \mathbf{w} be a fragment with support $\mathcal{I} \subset \mathcal{K}$. Suppose $\mathcal{I} = \mathcal{I}_0 \sqcup \mathcal{I}_1$, where $w_i = 0$ for all $i \in \mathcal{I}_0$ and $w_i = 1$ for all $i \in \mathcal{I}_1$. Then \mathbf{w} is $\mathcal{X}_{\mathfrak{C}}$ -forbidden if and only if $\mathcal{I}_0 \cap \text{conv}(\mathcal{I}_1) \neq \emptyset$. We can strip out all coordinates in $\mathcal{I}_0 \setminus \text{conv}(\mathcal{I}_1)$, and all but one of the coordinates in $\mathcal{I}_0 \cap \text{conv}(\mathcal{I}_1)$, and still have a $\mathcal{X}_{\mathfrak{C}}$ -forbidden fragment; hence we can assume that $\mathcal{I}_0 = \{k\}$ for some $k \in \text{conv}(\mathcal{I}_1)$. Next, find the smallest subset $\mathcal{J} \subseteq \mathcal{I}_1$ such that $k \in \text{conv}(\mathcal{J})$, and strip out all coordinates in $\mathcal{I}_1 \setminus \mathcal{J}$; the result is an $\mathcal{X}_{\mathfrak{C}}$ -critical fragment. At this point, we have $k \in \text{conv}(\mathcal{J})$ and $k \notin \text{conv}(\mathcal{J}')$ for all $\mathcal{J}' \subsetneq \mathcal{J}$. Thus, \mathcal{J} is Carathéodory-independent. \square

The next result is needed to prove Propositions 3.7, 4.5 and 6.2.

LEMMA A.2. *Let \mathfrak{C} be a convexity structure on \mathcal{K} . Then the associated convexity space $\mathcal{X}_{\mathfrak{C}}$ is McGarvey if and only if \mathfrak{C} contains all singletons.*

Proof: See Proposition 6.2(b) in [NP11]. \square

NOTATION. For any subset $\mathcal{I} \subseteq \mathcal{K}$, let $\mathbf{1}_{\mathcal{I}} \in \{0, 1\}^{\mathcal{I}}$ denote the \mathcal{I} -fragment which is equal to 1 in every coordinate of \mathcal{I} . (Note: the fragment $\mathbf{1}_{\mathcal{I}}$ is not the same as the view $\mathbf{1}^{\mathcal{I}}$.)

Proof of Proposition 3.7. The convexity space $\mathcal{X}_{\mathcal{E}}$ is McGarvey by Lemma A.2.

Thus, by Corollary 3.5, it suffices to show that $\text{Crit}(\mathcal{X}_{\mathcal{E}}) \neq \emptyset$. Let $\mathcal{J} \subseteq \mathcal{K}$ be a minimal subset such that $\text{conv}(\mathcal{J}) = \mathcal{K}$; we will show that $\mathbf{1}^{\mathcal{J}} \in \text{Crit}(\mathcal{X}_{\mathcal{E}})$. Let $k \in \mathcal{K}$, and let $\mathcal{I} \subseteq \mathcal{J}$; we say that \mathcal{I} is a \mathcal{J} -frame for k if $k \in \text{conv}(\mathcal{I})$, but $k \notin \text{conv}(\mathcal{H})$ for any proper subset $\mathcal{H} \subset \mathcal{I}$.

Let $k \in \mathcal{K} \setminus \mathcal{J}$ (so $\mathbf{1}_k^{\mathcal{J}} = 0$). By hypothesis, $k \in \text{conv}(\mathcal{J})$. Let $\mathcal{I} \subseteq \mathcal{J}$ be a \mathcal{J} -frame for k . Then \mathcal{I} is Carathéodory-independent (because $k \in \text{conv}(\mathcal{I})$ but $k \notin \text{conv}(\mathcal{H})$ for any $\mathcal{H} \subsetneq \mathcal{I}$). Thus, Lemma A.1 says the fragment $(\mathbf{1}_{\mathcal{I}}, 0_k)$ is critical. Clearly, $(\mathbf{1}_{\mathcal{I}}, 0_k) \sqsubset \mathbf{1}^{\mathcal{J}}$ and $k \in \text{supp}(\mathbf{1}_{\mathcal{I}}, 0_k)$. Thus, every $k \in \mathcal{K} \setminus \mathcal{J}$ is covered by some critical fragment compatible with $\mathbf{1}^{\mathcal{J}}$.

Next, let $j \in \mathcal{J}$ (so $\mathbf{1}_j^{\mathcal{J}} = 1$). Let $k \in \mathcal{K} \setminus \text{conv}(\mathcal{J} \setminus \{j\})$ (this set is nonempty precisely because \mathcal{J} is a *minimal* spanning set for \mathcal{K}). Let $\mathcal{I} \subseteq \mathcal{J}$ be a \mathcal{J} -frame for k . Then $\mathcal{I} \not\subseteq \mathcal{J} \setminus \{j\}$, because $k \notin \text{conv}(\mathcal{J} \setminus \{j\})$; thus, $j \in \mathcal{I}$. Just as in the previous paragraph, \mathcal{I} is Carathéodory-independent. Thus, Lemma A.1 says the fragment $(\mathbf{1}_{\mathcal{I}}, 0_k)$ is critical. Clearly, $(\mathbf{1}_{\mathcal{I}}, 0_k) \sqsubset \mathbf{1}^{\mathcal{J}}$ and $j \in \text{supp}(\mathbf{1}_{\mathcal{I}}, 0_k)$. Thus, every $j \in \mathcal{J}$ is covered by some critical fragment compatible with $\mathbf{1}^{\mathcal{J}}$.

The previous two paragraphs combined show that $\mathbf{1}^{\mathcal{J}} \in \text{Crit}(\mathcal{X}_{\mathcal{E}})$, as claimed. \square

Proof of Proposition 3.10. Let \mathbf{w} be an \mathcal{X} -critical fragment of size K . Then $\mathbf{w} \in \text{Crit}(\mathcal{X})$. Let $k_1, k_2, k_3 \in \mathcal{K}$ be three distinct coordinates. For $j = 1, 2, 3$, obtain \mathbf{x}^j from \mathbf{w} by negating w_{k_j} and leaving all the other coordinates the same. Then $\mathbf{x}^j \in \mathcal{X}$ (because \mathbf{w} is critical). Define $\mu \in \Delta_3^*(\mathcal{X})$ by $\mu[\mathbf{x}^j] = \frac{1}{3}$ for all $j \in \{1, 2, 3\}$. Then $\text{Maj}(\mu) = \mathbf{w}$; thus, μ is globally indeterminate, by Theorem 3.1. \square

The proof of Proposition 3.11 uses the following result:

LEMMA A.3. A profile $\mu \in \Delta^*(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$ is globally indeterminate if and only if the top cycle of μ is all of \mathcal{N} .

Proof: See Proposition 3.1(c) of [NPP14]. \square

Proof of Proposition 3.11. Without loss of generality, let $\mathcal{N} := [1 \dots N]$. For all $z \in \mathbb{Z}$, let $[z]$ be the unique element of \mathcal{N} such that $z \equiv [z] \pmod{N}$. (For example if $N = 7$ and $z = 11$, then $[z] = 4$.) Let $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$ represent any tournament ($\overset{\times}{\prec}$) such that,

$$(A2) \quad \forall n, m \in \mathcal{N}, \quad \left(m = [n + k] \text{ for some } k \in \mathbb{N} \text{ with } k < N/2 \right) \implies \left(n \overset{\times}{\succ} m \right).$$

(For example, if $N = 7$, then $5 \overset{\times}{\succ} 6$, $5 \overset{\times}{\succ} 7$, and $5 \overset{\times}{\succ} 1$, because $[6 - 5] = 1$, $[7 - 5] = 2$, and $[1 - 5] = 3$. However, $5 \overset{\times}{\succ} 2$, $5 \overset{\times}{\succ} 3$, and $5 \overset{\times}{\succ} 4$, because $[2 - 5] = 4$, $[3 - 5] = 5$, and $[4 - 5] = 6$; see Figure 2(a).)

If N is odd, then condition (A2) completely determines \mathbf{x} . If N is even, then for any $n, m \in \mathcal{N}$ with $m = [n + N/2]$, we also have $n = [m + N/2]$; in this case, we can set $n \overset{\times}{\succ} m$ or $m \overset{\times}{\succ} n$ arbitrarily. In any case, $\text{topcycle}(\mathbf{x}) = \mathcal{N}$. Thus, if $\text{Maj}(\mu) = \mathbf{x}$, then Lemma A.3 says that μ is globally indeterminate. Thus, it suffices to find a profile $\mu \in \Delta_3^*(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$ such that $\text{Maj}(\mu)$ satisfies condition (A2).

Let $N = 3q + r$ for some $q \in \mathbb{N}$ and $r \in \{0, 1, 2\}$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}_{\mathcal{N}}^{\text{pr}}$ correspond to the preference orders: $(1 \prec 2 \prec \dots \prec N)$, $((q + 1) \prec (q + 2) \prec \dots \prec N \prec$

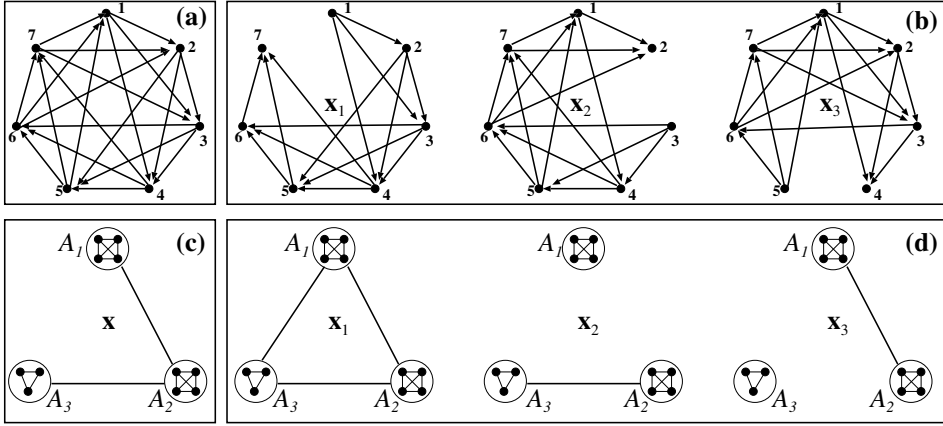


FIGURE 2. (a,b): The proof of Proposition 3.11, case $N = 7$. (c,d): The proof of Proposition 3.12.

$1 \prec 2 \prec \dots \prec (q-1) \prec q$ and $((2q+1) \prec (2q+2) \prec (2q+1) \prec \dots \prec N \prec 1 \prec 2 \prec \dots \prec (2q-1) \prec 2q)$, respectively [see Figure 2(b)]. Define $\mu \in \Delta_3^*(\mathcal{X}_N^{\text{pr}})$ by $\mu[\mathbf{x}_k] = \frac{1}{3}$ for $k = 1, 2, 3$. If $\mathbf{x} := \text{Maj}(\mu)$, then \mathbf{x} satisfies condition (A2), as desired. \square

The proof of Proposition 3.12 uses the following result:

LEMMA A.4. *Let \mathcal{K} be a subset of $\mathcal{N} \times \mathcal{N}$ containing exactly one of the pairs (n, m) or (m, n) for each $n \neq m \in \mathcal{N}$. Interpret each element of $\{0, 1\}^{\mathcal{K}}$ as an undirected graph, and define $\mathcal{X}_N^{\text{eq}} \subset \{0, 1\}^{\mathcal{K}}$ as in the third example of §3.3.*

Let $\mu \in \Delta^(\mathcal{X}_N^{\text{eq}})$, and consider the graph defined by $\text{Maj}(\mu)$. The profile μ is globally indeterminate if and only if this graph is connected, but not complete.*¹⁸

Proof: See Proposition 3.4(c) of [NPP14]. \square

Proof of Proposition 3.12. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \subset \mathcal{N}$ be three nonempty disjoint subsets such that $\mathcal{N} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$. Let $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$ represent the graph such that $n \sim m$ for all $n, m \in \mathcal{A}_i$ and $i \in \{1, 2, 3\}$. Furthermore $n \sim m$ for all $n \in \mathcal{A}_2$ and all $m \in \mathcal{A}_1 \sqcup \mathcal{A}_3$; however, $n \not\sim m$ for any $n \in \mathcal{A}_1$ and $m \in \mathcal{A}_3$ [see Figure 2(c)]. This graph is connected but not complete, so if $\text{Maj}(\mu) = \mathbf{x}$, then Lemma A.4 says μ is globally indeterminate. Thus, it suffices to find such a profile $\mu \in \Delta_3^*(\mathcal{X}_N^{\text{eq}})$.

Now, let \mathbf{x}_1 represent the complete equivalence relation $(\overset{1}{\sim})$ (i.e. $n \overset{1}{\sim} m$ for all $n, m \in \mathcal{N}$). Let $\mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}_N^{\text{eq}}$ represent the equivalence relations $(\overset{2}{\sim})$ and $(\overset{3}{\sim})$, described as follows [see Figure 2(d)]. First, we have $n \overset{i}{\sim} m$ for all $n, m \in \mathcal{A}_i$ and $i \in \{1, 2, 3\}$. Next, we have:

- $n \overset{2}{\sim} m$ for all $n \in \mathcal{A}_2$ and $m \in \mathcal{A}_3$.
- $n \overset{3}{\sim} m$ for all $n \in \mathcal{A}_1$ and $m \in \mathcal{A}_2$.

¹⁸Recall that a graph is *complete* if every node is connected to every other node by an edge, i.e. if the set of all nodes forms a clique.

Define $\mu \in \Delta_3^*(\mathcal{X}_{\mathcal{N}}^{\text{eq}})$ by $\mu[\mathbf{x}_n] = \frac{1}{3}$ for all $n \in \{1, 2, 3\}$. Then $\text{Maj}(\mu) = \mathbf{x}$, as desired. \square

The next result will be used to prove Proposition 3.13.

LEMMA A.5. *Let $M, D \in \mathbb{N}$, and let $\mu \in \Delta^*(\mathcal{X}_{M,D}^{\Delta})$. For each $d \in [1 \dots D]$, let $m_d^* := \text{med}_d(\mu)$ denote the **median** in coordinate d (that is: m_d^* is the unique $m \in [0 \dots M]$ such that $\mu(x_{d,m}) > \frac{1}{2} > \mu(x_{d,m+1})$).¹⁹ Let $D(\mu) := \left(\sum_{d=1}^D m_d^*\right) - M$. Then:*

- (a) *If $D(\mu) = 0$, then $\text{Cond}(\mathcal{X}_{M,D}^{\Delta}, \mu) = \text{Maj}(\mu)$.*
- (b) *If $D(\mu) > 0$, then $\text{Cond}(\mathcal{X}_{M,D}^{\Delta}, \mu) = \{\Phi(\mathbf{x}); \mathbf{x} \in \Delta_M^D \text{ and } x_d \in [m_d^* - D(\mu), m_d^*] \text{ for all } d \in [1 \dots D]\}$.*
- (c) *If $D(\mu) < 0$, then $\text{Cond}(\mathcal{X}_{M,D}^{\Delta}, \mu) = \{\Phi(\mathbf{x}); \mathbf{x} \in \Delta_M^D \text{ and } x_d \in [m_d^*, m_d^* + |D(\mu)|] \text{ for all } d \in [1 \dots D]\}$.*

Proof: See Proposition 3.9 of [NPP14]. \square

Proof of Proposition 3.13. For all $d \in \{1, 2, 3\}$, let \mathbf{x}^d be the element of $\mathcal{X}_{M,D}^{\Delta}$ which allocates all M dollars towards claimant d . (Thus, for all $m \in [1 \dots M]$, we have $x_{d,m}^d = 1$, while $x_{c,m}^d = 0$ for all $c \in [1 \dots D] \setminus \{d\}$.) Define $\mu \in \Delta_3^*(\mathcal{X}_{M,D}^{\Delta})$ by $\mu[\mathbf{x}_d] = \frac{1}{3}$ for $d = 1, 2, 3$. Then $\text{Maj}(\mu) = \mathbf{0}$, so $m_d^* = 0$ for all $d \in [1 \dots D]$, so that $D(\mu) = -M$. Thus, Lemma A.5(c) implies that $\text{Cond}(\mathcal{X}_{M,D}^{\Delta}, \mu) = \mathcal{X}_{M,D}^{\Delta}$, hence μ is globally indeterminate, by Theorem 3.1. \square

The following lemma will be useful in the proof of Proposition 3.14.

LEMMA A.6. *Consider the committee selection problem $\mathcal{X}_{I,J,K}^{\text{com}}$. For all $I \leq J$, we have $\text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}}) \subseteq \{\mathbf{0}, \mathbf{1}\}$. Moreover, if $I > 0$ then $\mathbf{0} \in \text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}})$, and if $J < K$ then $\mathbf{1} \in \text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}})$.*

Proof: The critical fragments of $\mathcal{X}_{I,J,K}^{\text{com}}$ are given as follows: if $I > 0$, then all fragments of exactly $K - I + 1$ zeros are critical; moreover, if $J < K$, then all fragments of exactly $J + 1$ ones are critical. (Example 2.4(b) illustrates these claims.) No other fragments are critical. This implies at once that $\mathbf{0} \in \text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}})$ if $I > 0$, and $\mathbf{1} \in \text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}})$ if $J < K$. Moreover, $I \leq J$, so $(K - I + 1) + (J + 1) > K$, so no element $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$ different from $\mathbf{0}$ and $\mathbf{1}$ can be critical for $\mathcal{X}_{I,J,K}^{\text{com}}$. \square

Proof of Proposition 3.14. (a) Let $I' := K - J$ and $J' := K - I$; then $I' \leq J'$.

Claim 1: $\eta(\mathcal{X}_{I',J',K}^{\text{com}}) = \eta(\mathcal{X}_{I,J,K}^{\text{com}})$.

Proof: For any $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$, define $\mathbf{x}' := (\neg x_k)_{k \in \mathcal{K}}$. For any $\mu \in \Delta(\{0, 1\}^{\mathcal{K}})$, define $\mu'(\mathbf{x}) := \mu(\mathbf{x}')$ for all $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$; then clearly $\text{Maj}(\mu') = \text{Maj}(\mu)'$. In particular, $\text{Maj}(\mu) = \mathbf{0}$ if and only if $\text{Maj}(\mu') = \mathbf{1}$. It is easy to check that $\mathcal{X}_{I',J',K}^{\text{com}} := \{\mathbf{x}'; \mathbf{x} \in \mathcal{X}_{I,J,K}^{\text{com}}\}$; thus, $\Delta_N^*(\mathcal{X}_{I',J',K}^{\text{com}}) := \{\mu'; \mu \in \Delta_N^*(\mathcal{X}_{I,J,K}^{\text{com}})\}$.

Lemma A.6 says that $\text{Crit}(\mathcal{X}_{I,J,K}^{\text{com}}) = \{\mathbf{0}, \mathbf{1}\} = \text{Crit}(\mathcal{X}_{I',J',K}^{\text{com}})$. Thus,

$$\begin{aligned} \eta(\mathcal{X}_{I,J,K}^{\text{com}}) &= \min \{N \in \mathbb{N}; \exists \mu \in \Delta_N^*(\mathcal{X}_{I,J,K}^{\text{com}}) \text{ with } \text{Maj}(\mu) = \mathbf{0} \text{ or } \mathbf{1}\} \\ &= \min \{N \in \mathbb{N}; \exists \mu \in \Delta_N^*(\mathcal{X}_{I',J',K}^{\text{com}}) \text{ with } \text{Maj}(\mu') = \mathbf{1} \text{ or } \mathbf{0}\} \\ &= \min \{N \in \mathbb{N}; \exists \mu' \in \Delta_N^*(\mathcal{X}_{I',J',K}^{\text{com}}) \text{ with } \text{Maj}(\mu') = \mathbf{1} \text{ or } \mathbf{0}\} \\ &= \eta(\mathcal{X}_{I',J',K}^{\text{com}}), \end{aligned}$$

¹⁹Such a value m_d^* exists because $\mu \in \Delta^*(\mathcal{X}_{M,D}^{\Delta})$.

as claimed. ◇ Claim 1

Now, $K - 2I' = 2J - K$ and $2J' - K = K - 2I$. Thus, $\min \left\{ \frac{K}{K-2I'}, \frac{K}{2J'-K} \right\} = \min \left\{ \frac{K}{K-2I}, \frac{K}{2J-K} \right\}$. This, together with Claim 1, means that the inequality (3.4) holds for $\mathcal{X}_{I,J;K}^{\text{com}}$, J and K if and only if it holds for $\mathcal{X}_{I',J';K}$, J' and K' ; thus, we can substitute one problem for the other. Furthermore, if $K - 2I < 2J - K$, then $K - 2I' \geq 2J' - K$. Thus, by exchanging $\mathcal{X}_{I',J';K}$ for $\mathcal{X}_{I,J;K}^{\text{com}}$ if necessary, we can assume without loss of generality that $K - 2I \geq 2J - K$. Since $I \leq J$, this implies $K - 2I \geq 0$; hence $K/2 \geq I$. Thus, $\eta(\mathcal{X}_{I,J;K}^{\text{com}}) \leq \min\{N \in \mathbb{N}; \exists \mu \in \Delta_N^*(\mathcal{X}_{I,J;K}^{\text{com}}) \text{ with } \text{Maj}(\mu) = \mathbf{0}\}$.

For all \mathbf{x} in the support of μ , we have $\sum_k x_k \geq I$; thus, $\sum_k \tilde{\mu}_k \geq I$. Thus, $\tilde{\mu}_k \geq I/K$, for some $k \in [1 \dots K]$. On the other hand, $\mathbf{0} = \text{Maj}(\mu)$ if and only if $\tilde{\mu}_k < \frac{1}{2}$ for all k . Thus, $\mathbf{0} = \text{Maj}(\mu)$ if and only if

$$\frac{I}{K} \leq \max_k \tilde{\mu}_k < \frac{1}{2}.$$

Suppose that the number of voters N is odd. For profiles μ to exist satisfying the latter inequality, one must have $\frac{N-1}{2N} \geq \frac{I}{K}$, i.e. $N \geq \frac{K}{K-2I}$ as claimed. (The case when N is even is similar.)

(b)[i] We first illustrate the argument in the case $\mathcal{X}_{4,6;11}^{\text{com}}$. In this case, $\frac{I}{K-2I} = \frac{4}{3} \in (1, 2]$, so $N = 2$, and the theorem claims that $\eta(\mathcal{X}_{4,6;11}^{\text{com}}) \leq 5$. We will construct a globally indeterminate profile with five voters. Define

$$\begin{aligned} \mathbf{x}_1 &:= (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0); \\ \mathbf{x}_2 &:= (0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0); \\ \mathbf{x}_3 &:= (1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1); \\ \mathbf{x}_4 &:= (0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0); \\ \text{and } \mathbf{x}_5 &:= (0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0). \end{aligned}$$

Define $\mu \in \Delta_5^*(\mathcal{X}_{4,6;11}^{\text{com}})$ by $\mu[\mathbf{x}_k] = \frac{1}{5}$ for $k = 1, \dots, 5$. Then clearly $\text{Maj}(\mu) = \mathbf{0} \in \text{Crit}(\mathcal{X}_{4,6;11}^{\text{com}})$. Thus, μ is globally indeterminate, by Theorem 3.1.

The general construction is similar. For any $z \in \mathbb{Z}$, let $[z]$ be the unique element of $[1 \dots K]$ such that $z \equiv [z] \pmod{K}$. Then, for all $n \in [0 \dots 2N+1]$, define $k_n := [nI]$. Next, for all $n \in [0 \dots 2N]$, let \mathcal{J}_n denote the ‘interval between’ k_n and $(k_{n+1}) - 1$ in a ‘mod K ’ sense. That is:

- if $1 \leq k_n \leq (k_{n+1}) - 1 \leq K$, then $\mathcal{J}_n := \{k_n, k_n + 1, \dots, (k_{n+1}) - 1\}$.
- if $1 \leq (k_{n+1}) - 1 \leq k_n \leq K$, then $\mathcal{J}_n := \{k_n, k_n + 1, \dots, K\} \cup \{1, 2, \dots, (k_{n+1}) - 1\}$.

Finally, for all $n \in [0 \dots 2N]$, let $\mathbf{x}^n := \mathbf{1}^{\mathcal{J}_n}$. Then $\mathbf{x}^n \in \mathcal{X}_{I,J;K}^{\text{com}}$ because $\|\mathbf{x}^n\| = |\mathcal{J}_n| = I$.

Now, define $\mu \in \Delta_{2N+1}^*(\mathcal{X}_{I,J;K}^{\text{com}})$ by $\mu[\mathbf{x}^n] = 1/(2N+1)$ for all $n \in [0 \dots 2N]$. We claim that $\text{Maj}(\mu) = \mathbf{0}$.

To see this, recall that $N := \left\lceil \frac{I}{K-2I} \right\rceil \geq \frac{I}{K-2I}$. Manipulating this inequality yields $(2N+1)I \leq NK$. This means that the sequence $k_0, k_1, \dots, k_{2N+1}$ ‘wraps around’ the interval $[1 \dots K]$ at most N times. This means that for each $k \in [1 \dots K]$, we have $|\{n \in [0 \dots 2N]; k \in \mathcal{J}_n\}| \leq N$. Thus, $\tilde{\mu}_k \leq N/(2N+1) < \frac{1}{2}$. Thus, $\text{Maj}_k(\mu) = 0$. This holds for all $k \in [1 \dots K]$, so $\text{Maj}(\mu) = \mathbf{0}$. But $\mathbf{0} \in \text{Crit}(\mathcal{X}_{I,J;K}^{\text{com}})$; thus, μ is globally indeterminate, by Theorem 3.1.

The proof of (b)[ii] is similar; simply reverse the roles of ‘0’ and ‘1’ in the proof of (b)[i]. \square

Proofs from Section 4.

Proof of Proposition 4.2. Lemma A.5 implies that μ is globally indeterminate only if $m_d^* = 0$ for all $d \in [1 \dots D]$, so that $D(\mu) = -M$. But this happens if and only if $\text{Maj}(\mu) = \mathbf{0}$. \square

Proof of Proposition 4.3. (a) follows immediately from Lemma 2.3(b), and (b) follows from (a). \square

Proof of Proposition 4.5. The space $\{\mathbf{1}^k\}_{k \in K} \cup \{\mathbf{1}\}$ is a convexity space; it is McGarvey by Lemma A.2. Moreover, any superset of a McGarvey space is clearly also McGarvey; thus, \mathcal{X} is McGarvey. It remains to show that \mathcal{X} does not admit a panopticon. By definition, a panopticon must lie outside \mathcal{X} . But $\mathbf{0}$ cannot be a panopticon for \mathcal{X} because any $\mathbf{1}^k$ is between $\mathbf{0}$ and $\mathbf{1}$. Thus consider $\mathbf{x} \notin \mathcal{X}$ with $\mathbf{x} \neq \mathbf{0}$. By assumption, \mathbf{x} must have at least two ones, say $x_\ell = 1$ and $x_m = 1$ with $\ell \neq m$, and at least one zero, say $x_h = 0$ for $h \notin \{\ell, m\}$. But then, the element $\mathbf{1}^\ell \in \mathcal{X}$ is between the elements \mathbf{x} and $\mathbf{1}^h \in \mathcal{X}$, i.e. \mathbf{x} is not a panopticon. \square

Proof of Proposition 4.7. First note that, if $|\mathcal{X}| = 1$, then trivially, \mathcal{X} is fully indeterminate. So assume that $|\mathcal{X}| \geq 2$. Let $\mu \in \Delta(\mathcal{X})$ be the uniform distribution on \mathcal{X} . (That is: $\mu(\mathbf{x}) = 1/|\mathcal{X}|$ for all $\mathbf{x} \in \mathcal{X}$.)

Claim 1: $\mu \in \Delta^*(\mathcal{X})$. Furthermore, if $\mathbf{z} := \text{Maj}(\mu)$, then $\gamma_*(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$.

Proof: Let $\gamma \in \Gamma_{\mathcal{X}}$. We have $\gamma_*[\mu] = \mu$, because \mathcal{X} is a finite set and $\gamma_* : \mathcal{X} \rightarrow \mathcal{X}$ is bijective. Thus, for every $k \in \mathcal{K}$, we have $\tilde{\mu}_k = \widetilde{\gamma_*[\mu]}_k = \tilde{\mu}_{\gamma(k)}$. However, $\mathcal{K} = \mathcal{K}_1 \sqcup \dots \sqcup \mathcal{K}_N$, a disjoint union of $\Gamma_{\mathcal{X}}$ -orbits. Thus, for any $n \in [1 \dots N]$, we have $\tilde{\mu}_k = \tilde{\mu}_j$ for all $j, k \in \mathcal{K}_n$. Thus, for any $n \in [1 \dots N]$, if $\tilde{\mu}_k \neq \frac{1}{2}$ for some $k \in \mathcal{K}_n$, then $\tilde{\mu}_k \neq \frac{1}{2}$ for all $k \in \mathcal{K}_n$.

By hypothesis, there is some $J \neq |\mathcal{K}_n|/2$ such that $\|\mathbf{x}_{\mathcal{K}_n}\| = J$ for some (and thus, all) $\mathbf{x} \in \mathcal{X}$. It follows that $\tilde{\mu}_k = J/|\mathcal{K}_n| \neq 1/2$ for some (and thus, all) $k \in \mathcal{K}_n$. This argument works for all $n \in [1 \dots N]$. Thus, $\mu \in \Delta^*(\mathcal{X})$.

Let $\mathbf{z} := \text{Maj}(\mu)$. For any $\gamma \in \Gamma_{\mathcal{X}}$, we have $\gamma_*(\mathbf{z}) = \mathbf{z}$, because $\tilde{\mu}_{\gamma(k)} = \tilde{\mu}_k$ for all $k \in \mathcal{K}$. \diamond **Claim 1**

Claim 2: $\mathbf{z} \notin \mathcal{X}$.

Proof: (by contradiction) Suppose $\mathbf{z} \in \mathcal{X}$. Then $\mathcal{X} = \{\mathbf{z}\}$, because \mathcal{X} is homogeneous, while Claim 1 says that $\gamma_*(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$. This contradicts our hypothesis that $|\mathcal{X}| \geq 2$. \diamond **Claim 2**

Claim 3: $\mathbf{z} \in \text{Pan}(\mathcal{X})$.

Proof: (by contradiction) Recall that $|\mathcal{X}| \geq 2$. For any distinct $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, write $\mathbf{x} < \mathbf{y}$ to mean that \mathbf{y} is between \mathbf{x} and \mathbf{z} (heuristic: \mathbf{y} is ‘closer’ than \mathbf{x} to \mathbf{z}).

Claim 3.1: For any distinct $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$ and permutation $\gamma : \mathcal{K} \rightarrow \mathcal{K}$, if $\mathbf{x} < \mathbf{y}$, then $\gamma_*(\mathbf{x}) < \gamma_*(\mathbf{y})$.

Proof: Fix $k \in \mathcal{K}$. If $\gamma_*(\mathbf{x})_k = z_k$ then $\gamma_*(\mathbf{x})_k = \gamma_*(\mathbf{z})_k$ (because $\mathbf{z} = \gamma_*(\mathbf{z})$ by Claim 1); hence $x_{\gamma(k)} = z_{\gamma(k)}$ (by definition of γ_*). Thus, $y_{\gamma(k)} = z_{\gamma(k)}$ (because $\mathbf{x} \prec \mathbf{y}$); and hence $\gamma_*(\mathbf{y})_k = \gamma_*(\mathbf{z})_k$ (by definition of γ_*), which means $\gamma_*(\mathbf{y})_k = z_k$ (because $\mathbf{z} = \gamma_*(\mathbf{z})$). This holds for all $k \in \mathcal{K}$; hence $\gamma_*(\mathbf{x}) \prec \gamma_*(\mathbf{y})$. ▽ Claim 3.1

Observe that (\prec) is a partial ordering on $\{0, 1\}^{\mathcal{K}}$ (it is transitive and antisymmetric). Also, \mathbf{z} is a panopticon for \mathcal{X} if and only if \mathcal{X} is a (\prec) -antichain.

So, suppose $\mathbf{x} \prec \mathbf{y}$ for some distinct $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Since \mathcal{X} is homogeneous, there is some $\gamma \in \Gamma_{\mathcal{X}}$ such that $\gamma(\mathbf{x}) = \mathbf{y}$. Thus, we have $\mathbf{x} \prec \gamma(\mathbf{x})$. Thus, Claim 2.1 says that $\gamma(\mathbf{x}) \prec \gamma^2(\mathbf{x})$, and then Claim 2.1 says that $\gamma^2(\mathbf{x}) \prec \gamma^3(\mathbf{x})$, and so on. But \mathcal{X} is finite, so there exists some $N \in \mathbb{N}$ such that $\gamma^N(\mathbf{x}) = \mathbf{x}$. Now we have $\mathbf{x} \prec \gamma(\mathbf{x}) \prec \gamma^2(\mathbf{x}) \prec \dots \prec \gamma^N(\mathbf{x}) = \mathbf{x}$. Thus, $\mathbf{x} \prec \mathbf{x}$ (because (\prec) is transitive). Contradiction.

Thus, there do not exist any distinct elements of \mathcal{X} which are related by (\prec) ; hence \mathbf{z} is a panopticon of \mathcal{X} , as desired. ◇ Claim 3

Now Proposition 4.3(a) implies that μ is fully indeterminate. □

REMARK. In Proposition 4.7, it is possible that, for some $n \in [1 \dots N]$, either all elements of \mathcal{X} are all-zeros on \mathcal{K}_n , or all elements of \mathcal{X} are all-ones on \mathcal{K}_n . (Indeed, the case $|\mathcal{X}| = 1$ occurs if and only if $\mathcal{X} = \{\mathbf{x}\}$ for some \mathbf{x} which is all-zeros or all-ones on \mathcal{K}_n for every $n \in [1 \dots N]$.) Surprisingly, this does not contradict full indeterminacy. But in such a case, the space will be fully indeterminate, but *not* globally indeterminate. This shows that neither full indeterminacy nor global indeterminacy is logically stronger than the other. (See also footnote 15.)

Proof of Proposition 4.9. Let μ be the uniform measure on \mathcal{X} . By an argument identical to Claim 1 in the proof of Proposition 4.7 (with $\mathcal{K}_1 = \mathcal{K}$ all being a single $\Gamma_{\mathcal{X}}$ -orbit), we deduce that $\mu \in \Delta^*(\mathcal{X})$; furthermore, if $\mathbf{z} := \text{Maj}(\mu)$, then $\gamma(\mathbf{z}) = \mathbf{z}$ for all $\gamma \in \Gamma_{\mathcal{X}}$. However, since $\Gamma_{\mathcal{X}}$ acts transitively on \mathcal{K} , this means that $z_k = z_j$ for all $j, k \in \mathcal{K}$. In other words, either $\mathbf{z} = \mathbf{0}$ or $\mathbf{z} = \mathbf{1}$.

Recall that $\|\mathbf{x}\| = C$ for all $\mathbf{x} \in \mathcal{X}$. Thus, $d(\mathbf{x}, \mathbf{0}) = C$ for all $\mathbf{x} \in \mathcal{X}$. From this, it follows that no $\mathbf{x} \in \mathcal{X}$ can be properly between $\mathbf{0}$ and any other $\mathbf{y} \in \mathcal{X}$. Thus, $\mathbf{0}$ is a panopticon for \mathcal{X} . Likewise, $d(\mathbf{x}, \mathbf{1}) = K - C$ for all $\mathbf{x} \in \mathcal{X}$, so $\mathbf{1}$ is a panopticon for \mathcal{X} . In either case, $\mathbf{z} = \text{Maj}(\mu)$ is a panopticon. Thus, Proposition 4.3(a) implies that μ is fully indeterminate. □

Proofs of results from Section 5. Proposition 5.1 follows immediately from the next lemma:

LEMMA A.7. *Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ and let $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$. The following are equivalent:*
(a) $\mathbf{z} \in \text{Pan}(\mathcal{X})$. **(b)** $I_{\mathbf{z}}[\mathcal{X}]$ is an antichain. **(c)** $\neg \mathbf{z} \in \text{Pan}(\mathcal{X})$.

Proof: For any $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, it is easy to check that $\mathbf{x} \asymp \mathbf{y}$ if and only if $I_{\mathbf{z}}(\mathbf{x}) \asymp I_{\mathbf{z}}(\mathbf{y})$. In particular, $\mathbf{x} \asymp \mathbf{z}$ if and only if $I_{\mathbf{z}}(\mathbf{x}) \asymp I_{\mathbf{z}}(\mathbf{z})$. But $I_{\mathbf{z}}(\mathbf{z}) = \mathbf{0}$. Thus, \mathbf{z} is a panopticon for \mathcal{X} if and only if $\mathbf{0}$ is a panopticon for $I_{\mathbf{z}}[\mathcal{X}]$.

Let $\mathcal{X}' := I_{\mathbf{z}}[\mathcal{X}]$, and let $\mathbf{x}, \mathbf{y} \in \mathcal{X}'$. Note that \mathbf{y} is between \mathbf{x} and $\mathbf{0}$ if and only if $\mathbf{y} \leq \mathbf{x}$. Thus, $\mathbf{0}$ is a panopticon for \mathcal{X}' if and only if \mathcal{X}' is an antichain. In this case, both $\mathbf{0}$ and $\mathbf{1}$ will be panoptica for \mathcal{X}' ; then both \mathbf{z} and $\neg \mathbf{z}$ are panoptica for \mathcal{X} . □

Proof of Proposition 5.2. (a) Combine Propositions 4.3(b) and 5.1.

(b) In Example 4.1, we saw that $\mathcal{X}_{J,J;K}^{\text{com}}$ is fully indeterminate. Clearly, $|\mathcal{X}_{J,J;K}^{\text{com}}| = \binom{K}{J}$. On the other hand, Sperner's Theorem says that any antichain in $\{0,1\}^K$ has cardinality at most $\binom{K}{J}$; see e.g. [Eng97, GK78]. \square

The proof of Proposition 5.3 requires some preliminaries.

LEMMA A.8. *Let \mathfrak{T} be a taxonomic hierarchy on \mathcal{K} . Let $\mathcal{J} \subset \mathcal{K}$. Then $\mathbf{1}^{\mathcal{J}}$ is a panopticon of $\mathcal{X}_{\mathfrak{T}}$ if and only if:*

- (a) *Either every minimal taxon of \mathfrak{T} intersects \mathcal{J} , or every minimal taxon of \mathfrak{T} intersects $\mathcal{J}^{\complement}$; and*
- (b) *Every non-minimal taxon of \mathfrak{T} contains at least two minimal taxa which intersect \mathcal{J} , and contains at least two minimal taxa which intersect $\mathcal{J}^{\complement}$.*

Proof: For any taxa $\mathcal{C}, \mathcal{D} \in \mathfrak{T}$, note that $\mathbf{1}^{\mathcal{D}}$ is strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$ if and only if

$$(A3) \quad (A3.1) \quad \mathcal{C} \cap \mathcal{J} \subseteq \mathcal{D} \cap \mathcal{J} \quad \text{and} \quad (A3.2) \quad \mathcal{C}^{\complement} \cap \mathcal{J}^{\complement} \subseteq \mathcal{D}^{\complement} \cap \mathcal{J}^{\complement},$$

and at least one of these containments is strict.

" \implies " Suppose $\mathbf{1}^{\mathcal{J}}$ is a panopticon of $\mathcal{X}_{\mathfrak{T}}$. Then condition (A3) can never be satisfied for any $\mathcal{C}, \mathcal{D} \in \mathfrak{T}$. We will use this to establish conditions (a) and (b). Condition (a) is equivalent to the following statement:

Claim 1: *There do not exist distinct minimal taxa $\mathcal{C}, \mathcal{D} \in \mathfrak{T}$ such that $\mathcal{C} \cap \mathcal{J} = \emptyset$ while $\mathcal{D} \cap \mathcal{J}^{\complement} = \emptyset$.*

Proof: (by contradiction) Suppose there exist minimal $\mathcal{C}, \mathcal{D} \in \mathfrak{T}$ such that $\mathcal{J} \cap \mathcal{C} = \emptyset$ and $\mathcal{J}^{\complement} \cap \mathcal{D} = \emptyset$. If $\mathcal{J}^{\complement} \cap \mathcal{D} = \emptyset$, then $\mathcal{D} \subseteq \mathcal{J}$, so $\mathcal{D} \cap \mathcal{J} = \mathcal{D}$. Thus, containment (A3.1) is strictly satisfied, because $\mathcal{J} \cap \mathcal{C} = \emptyset$.

Meanwhile, $\mathcal{D}^{\complement} \supseteq \mathcal{J}^{\complement}$, so $\mathcal{D}^{\complement} \cap \mathcal{J}^{\complement} = \mathcal{J}^{\complement}$. But if $\mathcal{J} \cap \mathcal{C} = \emptyset$, then $\mathcal{C} \subseteq \mathcal{J}^{\complement}$. Thus, $\mathcal{J}^{\complement} \cap \mathcal{C}^{\complement} = \mathcal{J}^{\complement} \setminus \mathcal{C} \subsetneq \mathcal{J}^{\complement}$ (because $\mathcal{C} \neq \emptyset$). Thus, containment (A3.2) is strictly satisfied. Thus, $\mathbf{1}^{\mathcal{D}}$ is strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. Contradiction.

\diamond Claim 1

To prove condition (b), we need the next result:

Claim 2: *For all non-minimal $\mathcal{C} \in \mathfrak{T}$, we have $\mathcal{C} \cap \mathcal{J} \neq \emptyset$ and $\mathcal{C} \cap \mathcal{J}^{\complement} \neq \emptyset$.*

Proof: (by contradiction) Let $\mathcal{C} \in \mathfrak{T}$ be non-minimal; then there exists some $\mathcal{D} \in \mathfrak{T}$ with $\mathcal{D} \subsetneq \mathcal{C}$.

First, suppose $\mathcal{J} \cap \mathcal{C} = \emptyset$; then we have $\mathcal{D} \subsetneq \mathcal{C} \subseteq \mathcal{J}^{\complement}$. Thus, $\mathcal{D} \cap \mathcal{J} = \emptyset = \mathcal{C} \cap \mathcal{J}$, so containment (A3.1) is (weakly) satisfied. Meanwhile $\mathcal{D}^{\complement} \cap \mathcal{J}^{\complement} = \mathcal{J}^{\complement} \setminus \mathcal{D} \supsetneq \mathcal{J}^{\complement} \setminus \mathcal{C} = \mathcal{C}^{\complement} \cap \mathcal{J}^{\complement}$, so containment (A3.2) is strictly satisfied. Thus, $\mathbf{1}^{\mathcal{D}}$ is strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. Contradiction.

Now suppose $\mathcal{J}^{\complement} \cap \mathcal{C} = \emptyset$. Then we have $\mathcal{D} \subsetneq \mathcal{C} \subseteq \mathcal{J}$. Thus $\mathcal{D} \cap \mathcal{J} = \mathcal{D} \subsetneq \mathcal{C} = \mathcal{C} \cap \mathcal{J}$, so the reverse of containment (A3.1) is strictly satisfied. Meanwhile $\mathcal{D}^{\complement} \supset \mathcal{C}^{\complement} \supseteq \mathcal{J}^{\complement}$, so $\mathcal{D}^{\complement} \cap \mathcal{J}^{\complement} = \mathcal{J}^{\complement} = \mathcal{C}^{\complement} \cap \mathcal{J}^{\complement}$, so the reverse of containment (A3.2) is (weakly) satisfied. Thus, $\mathbf{1}^{\mathcal{D}}$ is strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. Contradiction. \diamond Claim 2

To prove condition (b), note that, if \mathcal{C} is any non-minimal taxon, then \mathcal{C} is a disjoint union of two or more minimal taxa.

At least two of these taxa intersect \mathcal{J} . (by contradiction) If none of the minimal taxa in \mathcal{C} intersects \mathcal{J} , then $\mathcal{C} \cap \mathcal{J} = \emptyset$, contradicting Claim 2. Suppose only one of the minimal taxa inside \mathcal{C} intersects \mathcal{J} —call it \mathcal{D} . Then $\mathcal{C} \cap \mathcal{J} = \mathcal{D} \cap \mathcal{J}$, so containment (A3.1) is weakly satisfied. Meanwhile, $\mathcal{D}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}} = (\mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}}) \sqcup (\mathcal{C} \setminus \mathcal{D}) \supsetneq (\mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}})$, so containment (A3.2) is strictly satisfied. Thus, $\mathbf{1}^{\mathcal{D}}$ is strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. Contradiction.

At least two of these taxa intersect $\mathcal{J}^{\mathcal{C}}$. (by contradiction) If none of the minimal taxa in \mathcal{C} intersect $\mathcal{J}^{\mathcal{C}}$, then $\mathcal{C} \cap \mathcal{J}^{\mathcal{C}} = \emptyset$, contradicting Claim 2. Suppose only one of the minimal taxa in \mathcal{C} intersects $\mathcal{J}^{\mathcal{C}}$ —call it \mathcal{D} . Then $\mathcal{C} \cap \mathcal{J} = (\mathcal{D} \cap \mathcal{J}) \sqcup (\mathcal{C} \setminus \mathcal{D}) \supsetneq (\mathcal{D} \cap \mathcal{J})$, so the reverse of containment (A3.1) is strictly satisfied. Meanwhile, $\mathcal{D}^{\mathcal{C}} = \mathcal{C}^{\mathcal{C}} \sqcup (\mathcal{C} \setminus \mathcal{D})$, so $\mathcal{D}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}} = (\mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}}) \sqcup ((\mathcal{C} \setminus \mathcal{D}) \cap \mathcal{J}^{\mathcal{C}}) = \mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}}$, because $(\mathcal{C} \setminus \mathcal{D}) \cap \mathcal{J}^{\mathcal{C}} = \emptyset$. Thus, the reverse of containment (A3.2) is weakly satisfied. Thus, $\mathbf{1}^{\mathcal{C}}$ is strictly between $\mathbf{1}^{\mathcal{D}}$ and $\mathbf{1}^{\mathcal{J}}$. Contradiction.

“ \Leftarrow ” Suppose \mathcal{J} satisfies the conditions (a) and (b). To show that $\mathbf{1}^{\mathcal{J}}$ is a panopticon for $\mathcal{X}_{\mathfrak{T}}$, we must show, for any taxon $\mathcal{C} \in \mathfrak{T}$, that $\mathbf{1}^{\mathcal{C}} \asymp \mathbf{1}^{\mathcal{J}}$. Let $\mathcal{D} \in \mathfrak{T}$; we will show that $\mathbf{1}^{\mathcal{D}}$ cannot be strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. Since \mathfrak{T} is a taxonomic hierarchy, there are only three cases: either $\mathcal{D} \subset \mathcal{C}$, or $\mathcal{C} \subset \mathcal{D}$, or \mathcal{D} is disjoint from \mathcal{C} .

Case 1. Suppose $\mathcal{D} \subset \mathcal{C}$. Then \mathcal{C} must be non-minimal, so condition (b) says that \mathcal{C} contains some other taxon \mathcal{B} besides \mathcal{D} , such that $\mathcal{B} \cap \mathcal{J} \neq \emptyset$. This means that $\mathcal{C} \cap \mathcal{J} \supsetneq \mathcal{D} \cap \mathcal{J}$. Thus, containment (A3.1) is falsified.

Case 2. Suppose $\mathcal{C} \subset \mathcal{D}$. Then \mathcal{D} must be non-minimal, so condition (b) says that \mathcal{D} contains some other taxon \mathcal{B} besides \mathcal{C} , such that $\mathcal{B} \cap \mathcal{J}^{\mathcal{C}} \neq \emptyset$. This means that $\mathcal{D} \cap \mathcal{J}^{\mathcal{C}} \supsetneq \mathcal{C} \cap \mathcal{J}^{\mathcal{C}}$, which means $\mathcal{D}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}} \subsetneq \mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}}$. Thus, containment (A3.2) is falsified.

Case 3. Suppose \mathcal{D} is disjoint from \mathcal{C} . Then $\mathcal{D} \cap \mathcal{J}$ is disjoint from $\mathcal{C} \cap \mathcal{J}$, so if $\mathcal{C} \cap \mathcal{J} \neq \emptyset$, then containment (A3.1) is falsified.

If $\mathcal{C} \cap \mathcal{J} = \emptyset$, then condition (b) implies that \mathcal{C} must be minimal. But then condition (a) implies that *every* taxon in \mathfrak{T} intersects $\mathcal{J}^{\mathcal{C}}$. In particular, $\mathcal{D} \cap \mathcal{J}^{\mathcal{C}} \neq \emptyset$. But $\mathcal{D} = \mathcal{D} \cap \mathcal{C}^{\mathcal{C}}$ (because $\mathcal{D} \subset \mathcal{C}^{\mathcal{C}}$, because \mathcal{C} and \mathcal{D} are disjoint). Thus, we have $\mathcal{D} \cap \mathcal{C}^{\mathcal{C}} \cap \mathcal{J}^{\mathcal{C}} \neq \emptyset$. Thus, $\mathcal{D}^{\mathcal{C}} \not\supseteq \mathcal{J}^{\mathcal{C}} \cap \mathcal{C}^{\mathcal{C}}$, so containment (A3.2) is falsified.

In any case, one of the conditions in (A3) fails, so $\mathbf{1}^{\mathcal{D}}$ cannot be strictly between $\mathbf{1}^{\mathcal{C}}$ and $\mathbf{1}^{\mathcal{J}}$. This holds for all $\mathcal{D} \in \mathfrak{T}$. Thus, $\mathbf{1}^{\mathcal{C}} \asymp \mathbf{1}^{\mathcal{J}}$. \square

Proof of Proposition 5.3. (a) “ \implies ” (by contradiction) If $\mathcal{X}_{\mathfrak{T}}$ is a generalized antichain, then it has a panopticon, say, $\mathbf{1}^{\mathcal{J}}$, which must satisfy conditions (a) and (b) of Lemma A.8. Now, any singleton taxon is minimal, and any singleton is either contained in \mathcal{J} or in $\mathcal{J}^{\mathcal{C}}$. Thus, condition (a) of Lemma A.8 says that either (a1) \mathcal{J} contains *all* the singleton taxa of \mathfrak{T} , or (a2) \mathcal{J} contains *none* of the singleton taxa of \mathfrak{T} .

Now, let $\mathcal{C} \in \mathfrak{T}$ be non-minimal. If \mathcal{C} does *not* contain two non-singleton minimal taxa, then either (a1) \mathcal{C} does not contain two minimal taxa intersecting $\mathcal{J}^{\mathcal{C}}$, or (a2) \mathcal{C} does not contain two minimal taxa intersecting \mathcal{J} . Either way, condition (b) of Lemma A.8 is falsified, yielding a contradiction.

(a) “ \Leftarrow ” Let $\mathcal{J} \subset \mathcal{K}$ contain exactly *one* representative from every minimal taxon in \mathfrak{T} . To show that $\mathbf{1}^{\mathcal{J}}$ is a panopticon, we must check conditions

(a) and (b) of Lemma A.8. By construction, every minimal taxon intersects \mathcal{J} ; hence the first alternative of (a) is satisfied.

To see (b), suppose \mathcal{C} is non-minimal. By hypothesis, \mathcal{C} contains at least two non-singleton minimal taxa, each of which contains exactly one point in \mathcal{J} —thus each of them intersects both \mathcal{J} and \mathcal{J}^c .

Thus Lemma A.8 says that $\mathbf{1}^{\mathcal{J}}$ is a panopticon.

(b) We will show that $\text{Maj}(\mathcal{X}_{\mathfrak{T}})$ contains no panopticon points. Let $\mathcal{J} \subseteq \mathcal{K}$. If $\mathbf{1}^{\mathcal{J}} \in \text{Maj}(\mathcal{X}_{\mathfrak{T}})$, then \mathcal{J} is a union of taxa of \mathfrak{T} [NP11, Proposition 6.2(a)]. Every taxon in \mathfrak{T} is a disjoint union of minimal taxa. Thus, \mathcal{J} is a disjoint union of minimal taxa of \mathfrak{T} . Thus, if $\mathcal{C} \in \mathfrak{T}$ is a minimal taxon, then we have:

$$(A4) \quad (\mathcal{C} \cap \mathcal{J} \neq \emptyset) \implies (\mathcal{C} \subseteq \mathcal{J}) \quad \text{and} \quad (\mathcal{C} \cap \mathcal{J}^c \neq \emptyset) \implies (\mathcal{C} \subseteq \mathcal{J}^c).$$

By contradiction, suppose $\mathbf{1}^{\mathcal{J}}$ was a panopticon for $\mathcal{X}_{\mathfrak{T}}$. Then Lemma A.8(a) says that either \mathcal{J} intersects every minimal taxon, or \mathcal{J}^c intersects every minimal taxon. But then statement (A4) says that either \mathcal{J} entirely *includes* every minimal taxon, or \mathcal{J}^c entirely includes every minimal taxon. That is: either $\mathcal{J} = \mathcal{K}$, or $\mathcal{J} = \emptyset$. But in either case, Lemma A.8(b) is contradicted. Thus, \mathcal{J} cannot be a panopticon. \square

Proof of Proposition 5.4(a). (by contradiction) Suppose $\mathbf{y} \in \mathcal{X}$ is adjacent to \mathbf{x} . Then for any $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$, $I_{\mathbf{z}}(\mathbf{y})$ is adjacent to $I_{\mathbf{z}}(\mathbf{x})$. But then either $I_{\mathbf{z}}(\mathbf{y}) \leq I_{\mathbf{z}}(\mathbf{x})$ or $I_{\mathbf{z}}(\mathbf{y}) \geq I_{\mathbf{z}}(\mathbf{x})$; hence $I_{\mathbf{z}}(\mathcal{X})$ is not an antichain. Contradiction. \square

The proof of Proposition 5.4(b) requires some preliminaries. For any $\mathbf{x}, \mathbf{y} \in \{0, 1\}^{\mathcal{K}}$, define $\mathbf{x} \setminus \mathbf{y} := \mathbf{x} \wedge (\neg \mathbf{y})$. That is, for all $k \in \mathcal{K}$, $(\mathbf{x} \setminus \mathbf{y})_k = 1$ if and only if $x_k = 1$ and $y_k = 0$. Now define

$$\begin{aligned} \mathcal{Z}(\mathbf{x}, \mathbf{y}) &:= \{ \mathbf{z} \in \{0, 1\}^{\mathcal{K}} ; \mathbf{z} \geq (\mathbf{x} \setminus \mathbf{y}) \text{ and } \neg \mathbf{z} \geq (\mathbf{y} \setminus \mathbf{x}) \} \\ &= \left\{ \mathbf{z} \in \{0, 1\}^{\mathcal{K}} ; \begin{array}{l} \forall k \in \mathcal{K}, (x_k = 1 \text{ and } y_k = 0) \implies (z_k = 1) \\ \text{and } (x_k = 0 \text{ and } y_k = 1) \implies (z_k = 0) \end{array} \right\}. \end{aligned}$$

Finally, for any $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$, define $\mathcal{Z}(\mathcal{X}) := \bigcup_{\mathbf{x} \neq \mathbf{y} \in \mathcal{X}} \mathcal{Z}(\mathbf{x}, \mathbf{y}) \subseteq \{0, 1\}^{\mathcal{K}}$.

LEMMA A.9. (a) For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{0, 1\}^{\mathcal{K}}$,

$$(I_{\mathbf{z}}(\mathbf{x}) \leq I_{\mathbf{z}}(\mathbf{y})) \iff (\mathbf{z} \in \mathcal{Z}(\mathbf{x}, \mathbf{y})).$$

(b) $|\mathcal{Z}(\mathbf{x}, \mathbf{y})| = 2^{K - d_H(\mathbf{x}, \mathbf{y})}$.

(c) Let $\mathcal{X} \subseteq \{0, 1\}^{\mathcal{K}}$ and let $\mathcal{Z}(\mathcal{X})^c := \{0, 1\}^{\mathcal{K}} \setminus \mathcal{Z}(\mathcal{X})$. For any $\mathbf{z} \in \{0, 1\}^{\mathcal{K}}$, we have:

$$(\mathbf{z} \in \mathcal{Z}(\mathcal{X})^c) \iff (I_{\mathbf{z}}(\mathcal{X}) \text{ is an antichain}).$$

Thus \mathcal{X} is a generalized antichain if and only if $\mathcal{Z}(\mathcal{X}) \neq \{0, 1\}^{\mathcal{K}}$.

Proof: (a) is clear from the definition of $\mathcal{Z}(\mathbf{x}, \mathbf{y})$. (b) is because $\mathcal{Z}(\mathbf{x}, \mathbf{y})$ is defined by $\|\mathbf{x} \setminus \mathbf{y}\| + \|\mathbf{y} \setminus \mathbf{x}\| = d_H(\mathbf{x}, \mathbf{y})$ constraints, leaving $K - d_H(\mathbf{x}, \mathbf{y})$ free coordinates.

$$(c) \left(\mathbf{z} \in \mathcal{Z}(\mathcal{X})^0 \right) \iff \left(\mathbf{z} \notin \mathcal{Z}(\mathbf{x}, \mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathcal{X} \right) \xLeftrightarrow{(\dagger)} \left(I_{\mathbf{z}}(\mathbf{x}) \not\leq I_{\mathbf{z}}(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathcal{X} \right) \\ \iff \left(I_{\mathbf{z}}(\mathcal{X}) \text{ is an antichain} \right). \text{ Here, } (\dagger) \text{ is by part (a).} \quad \square$$

Proof of Proposition 5.4(b). Lemma A.9(c) implies that \mathcal{X} is a generalized antichain if and only if $|\mathcal{Z}(\mathcal{X})| < 2^K$. But

$$|\mathcal{Z}(\mathcal{X})| \leq \sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{X}} |\mathcal{Z}(\mathbf{x}, \mathbf{y})| \stackrel{(*)}{=} \sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{X}} 2^{K-d_H(\mathbf{x}, \mathbf{y})} = 2^K \sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{X}} 2^{-d_H(\mathbf{x}, \mathbf{y})},$$

where $(*)$ is by Lemma A.9(b). \square

Proof of Proposition 5.5. Let $\mathcal{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N\}$, and fix some $m \in [1 \dots N]$. The Law of Large Numbers says that

$$\frac{1}{N} \sum_{\substack{n=1 \\ n \neq m}}^N 2^{-d_H(\mathbf{x}_n, \mathbf{x}_m)} \approx \mathbb{E} \left[2^{-d_H(\mathbf{y}, \mathbf{x}_m)} \mid \mathbf{y} \in \{0, 1\}^K \text{ random and uniformly distributed} \right] \\ = \mathbb{E} \left[2^{-d_H(\mathbf{y}, \mathbf{0})} \mid \mathbf{y} \in \{0, 1\}^K \text{ random and uniformly distributed} \right] \\ = \frac{1}{2^K} \sum_{L=0}^K \binom{K}{L} 2^{-L} = \frac{1}{2^K} \left(1 + \frac{1}{2}\right)^K = \frac{1}{2^K} \left(\frac{3}{2}\right)^K = \left(\frac{3}{4}\right)^K.$$

$$\text{Thus, we expect that } \sum_{m=1}^N \sum_{\substack{n=1 \\ n \neq m}}^N 2^{-d_H(\mathbf{x}_n, \mathbf{x}_m)} \approx N^2 \cdot \left(\frac{3}{4}\right)^K < 1,$$

where the last step is because we assume $N < \left(\frac{4}{3}\right)^{K/2}$. As $K \rightarrow \infty$, the probability of strict inequality increases to 1, and in this case, Proposition 5.4(b) implies that \mathcal{X} is a generalized antichain. \square

Proofs from Section 6. We will use the following simple but elegant result of [Sze43]. For completeness, we include the proof.

LEMMA A.10. *The expected number of directed Hamiltonian paths which exist in a randomly generated tournament (where all edges are independent random variables with both orientations having equal probability) is given by $\frac{N!}{2^{N-1}}$.*

Proof: There are $N!$ directed Hamiltonian chains through \mathcal{N} . For any such chain, and any random tournament, there is a probability $1/2^{N-1}$ that the chain can be embedded in the tournament (because each of the $N-1$ edges of the chain has probability $1/2$ of being compatible with the corresponding edge in the tournament, and these $N-1$ events are all independent random variables). \square

Proof of Proposition 6.1. (a) Lemma A.10 implies that there exists a tournament on \mathcal{N} (i.e. an element $\mathbf{x} \in \{0, 1\}^K$) with at least $\frac{N!}{2^{N-1}}$ distinct Hamiltonian chains. The theorem of [McG53] says that there is some $\mu \in \Delta(\mathcal{X}_{\mathcal{N}}^{\text{pr}})$ with

$\text{Maj}(\mu) = \mathbf{x}$; then Proposition 2.6(a) says that $|\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)| \geq \frac{N!}{2^{N-1}}$. Thus,

$$\begin{aligned} h(\mathcal{X}_{\mathcal{N}}^{\text{pr}}) &\geq h(\mu) = \frac{\log_2 |\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{pr}}, \mu)|}{\log_2 |\mathcal{X}_{\mathcal{N}}^{\text{pr}}|} \geq \frac{\log_2(N!) - \log_2(2^{N-1})}{\log_2(N!)} \\ &= 1 - \frac{N-1}{\log_2(N!)} \\ &\stackrel{(*)}{\approx} 1 - \frac{N-1}{(N + \frac{1}{2}) \log_2(N) - N \log_2(e) + \log_2(\sqrt{2\pi})} \\ &\approx 1 - \frac{1}{\log_2(N)} \xrightarrow{N \rightarrow \infty} 1. \end{aligned}$$

Here (*) is because Stirling's formula says that $N! \approx \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$.

(b) Fix $c \in (0, 1)$. Let $L := \lfloor cN \rfloor$ and let $R := N - L$. Let $\mathbf{x} \in \{0, 1\}^{\mathcal{K}}$ represent the complete bipartite graph which has L 'left' vertices and R 'right' vertices, where every left vertex is linked to every right vertex (but there are no links between any two left vertices or any two right vertices). The space $\mathcal{X}_{\mathcal{N}}^{\text{eq}}$ is McGarvey [NP11, Example 3.9(a)], so there exists some $\mu \in \Delta^*(\mathcal{X}_{\mathcal{N}}^{\text{eq}})$ such that $\text{Maj}(\mu) = \mathbf{x}$. Let $\mathcal{Y} \subseteq \mathcal{X}_{\mathcal{N}}^{\text{eq}}$ be the set of all equivalence relations defined as follows:

- Fix $M \in [1 \dots L]$. Partition the set of right-hand vertices into exactly M disjoint subsets R_1, R_2, \dots, R_M (some of which may be empty).
- Let the left-hand vertices be v_1, v_2, \dots, v_L . For all $m \in [1 \dots M]$, declare every element of R_m to be equivalent to v_m and equivalent to every other element of R_m .
- Declare v_M, \dots, v_L to be equivalent to one another and equivalent to every element of R_M .

Claim 1: $\mathcal{Y} \subseteq \text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{eq}}, \mu)$.

Proof: Given any $\mathbf{y} \in \mathcal{Y}$, we will construct a path ζ such that $F^\zeta(\mu) = \mathbf{y}$. We do this as follows:

- (1) For each $m \in [1 \dots M]$, and each vertex $r \in R_m$, the path ζ first visits the coordinate (v_m, r) ; in every one of these coordinates, the majority prevails, so we get $F_{(v_m, r)}^\zeta(\mu) = x_{(v_m, r)} = 1 = y_{(v_m, r)}$ (encoding the equivalence $v_m \sim r$).
- (2) At this point, for all $m \in [1 \dots M]$ and all $r, r' \in R_m$, transitivity constraints force $r \sim r'$ —i.e. we get $F_{(r, r')}^\zeta(\mu) = 1 = y_{(r, r')}$.
- (3) Next, for each $m, \ell \in [1 \dots M]$ with $m \neq \ell$, the path ζ visits the coordinate (v_m, v_ℓ) . Again, the majority prevails, so $F_{(v_m, v_\ell)}^\zeta(\mu) = x_{(v_m, v_\ell)} = 0 = y_{(v_m, v_\ell)}$ (encoding the nonequivalence $v_m \not\sim v_\ell$).
- (4) At this point, for all $m, \ell \in [1 \dots M]$ with $m \neq \ell$, and all $r \in R_m$ and $r' \in R_\ell$, transitivity constraints force $r \not\sim r'$ —i.e. we get $F_{(r, r')}^\zeta(\mu) = 0 = y_{(r, r')}$.
- (5) Next, fix some $r_0 \in R_M$. For each $\ell \in [M \dots L]$, the path ζ visits the coordinate (r_0, v_ℓ) . Again, the majority prevails, so $F_{(r_0, v_\ell)}^\zeta(\mu) = x_{(r_0, v_\ell)} = 1 = y_{(r_0, v_\ell)}$ (encoding the equivalence $r_0 \sim v_\ell$).

- (6) At this point, for all $\ell, m \in [M \dots L]$ transitivity constraints force $v_\ell \sim v_m$ —i.e. we get $F_{(v_\ell, v_m)}^\zeta(\mu) = 1 = y_{(v_\ell, v_m)}$ (encoding the equivalence $v_\ell \sim v_m$).
- (7) Finally, visit the remaining elements of \mathcal{K} in some arbitrary order. (The values for these coordinates are already completely determined by the transitivity constraints in steps 2, 4, and 6).

At this point, we have $F_k^\zeta(\mu) = y_k$ for all $k \in \mathcal{K}$, as desired. Thus, $F^\zeta(\mu) = \mathbf{y}$. Thus, $\mathbf{y} \in \text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{eq}}, \mu)$, by statement (2.1). \diamond claim 1

Claim 2: $|\mathcal{Y}| \geq L^{R+1}/(R+1)$.

Proof: For any fixed $M \in [1 \dots L]$, let \mathcal{Y}_M be the elements of \mathcal{Y} obtained by partitioning the right-hand vertices into M subsets (labelled by v_1, \dots, v_M , and some possibly empty). Then $|\mathcal{Y}_M| = M^R$ (because any such partition corresponds to a function from the R right-hand vertices into $\{v_1, \dots, v_M\}$). Now, $\mathcal{Y} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup \dots \sqcup \mathcal{Y}_L$, so

$$\begin{aligned} |\mathcal{Y}| &= |\mathcal{Y}_1| + |\mathcal{Y}_2| + \dots + |\mathcal{Y}_L| = 1^R + 2^R + \dots + L^R \\ &\geq \int_0^L x^R dx = \frac{L^{R+1}}{R+1}, \end{aligned}$$

as claimed

\diamond Claim 2

Claim 3: $|\mathcal{X}_{\mathcal{N}}^{\text{eq}}| \leq N^N$.

Proof: Any element of $\mathcal{X}_{\mathcal{N}}^{\text{eq}}$ can be represented by a partition of $[1 \dots N]$ into N unlabelled subsets (some of which may be empty). We can represent a *labelled* partition by a function $f : [1 \dots N] \rightarrow [1 \dots N]$. There are N^N such functions, and hence N^N labelled partitions. (Of course, many of these labelled partitions correspond to the same *unlabelled* partition.) Thus, $|\mathcal{X}_{\mathcal{N}}^{\text{eq}}| \leq N^N$. \diamond claim 3

In light of these three claims, we have

$$\begin{aligned} h(\mathcal{X}_{\mathcal{N}}^{\text{eq}}) &\geq \frac{\log_2 |\text{Cond}(\mathcal{X}_{\mathcal{N}}^{\text{eq}}, \mu)|}{\log_2 |\mathcal{X}_{\mathcal{N}}^{\text{eq}}|} \stackrel{(*)}{\geq} \frac{\log_2(L^{R+1}/(R+1))}{\log_2(N^N)} \\ &= \frac{(R+1)\log_2(L) - \log_2(R+1)}{N \log_2(N)} \\ &\stackrel{(\dagger)}{\approx} \frac{[(1-c)N+1]\log_2(cN) - \log_2((1-c)N+1)}{N \log_2(N)} \\ &\approx \frac{[(1-c)N+1][\log_2(N) + \log(c)] - \log_2(1-c) - \log_2(N)}{N \log_2(N)} \\ &\xrightarrow{N \rightarrow \infty} (1-c). \end{aligned}$$

Here, $(*)$ is by Claims 1, 2 and 3, while (\dagger) is because $L = \lfloor cN \rfloor \approx cN$, so that $R = N - L \approx N - cN = (1-c)N$.

However, c can be any value in $(0, 1)$. By letting $c \rightarrow 0$, we conclude that $h(\mathcal{X}_{\mathcal{N}}^{\text{eq}}) \geq 1$; hence $h(\mathcal{X}_{\mathcal{N}}^{\text{eq}}) = 1$, as desired. \square

Proof of Proposition 6.2. (a) Let $\mathcal{K} = [1 \dots K]$, and let $\mathfrak{C}_K^{\text{line}}$ be the linear graph convexity on \mathcal{K} . The convex subsets of $\mathfrak{C}_K^{\text{line}}$ are the intervals $[j \dots k]$ for any $j \leq k \in \mathcal{K}$. Let $\mathcal{J} := \{k \in \mathcal{K}; k \text{ odd}\}$. Note that $\mathbf{1}^{\mathcal{J}} \in \text{Maj}(\mathfrak{C}_K^{\text{line}})$ because $\mathcal{X}_K^{\text{line}}$ is McGarvey by Lemma A.2. For any $j \leq k \in \mathcal{K}$, we have $\mathbf{1}^{[j \dots k]} \asymp \mathbf{1}^{\mathcal{J}}$ if and

only if j and k are both odd. Thus, $|\mathcal{X}_K^{\text{line}}(\mathbf{1}^{\mathcal{J}})| \geq \frac{1}{4}|\mathcal{X}_K^{\text{line}}|$ (since at least one quarter of the subintervals of \mathcal{K} have two odd endpoints). Thus,

$$h(\mathcal{X}_K^{\text{line}}) \geq \frac{\log_2(\frac{1}{4}|\mathcal{X}_K^{\text{line}}|)}{\log_2|\mathcal{X}_K^{\text{line}}|} = \frac{\log_2|\mathcal{X}_K^{\text{line}}| - 2}{\log_2|\mathcal{X}_K^{\text{line}}|} \xrightarrow{K \rightarrow \infty} 1.$$

Thus, $\mathcal{X}_K^{\text{line}}$ is asymptotically fully indeterminate.

(b) Let $D \in \mathbb{N}$, let $K = 2^D$, and let $\varphi : \mathcal{K} \rightarrow \{0, 1\}^D$ be some bijection. Let \mathfrak{C}_D^{\square} be the hypercube convexity on $\{0, 1\}^D$, and define $\Phi : \mathcal{P}(\{0, 1\}^D) \rightarrow \{0, 1\}^{\mathcal{K}}$ by $\Phi(\mathcal{C}) := \mathbf{1}^{\varphi^{-1}(\mathcal{C})}$ for all $\mathcal{C} \subseteq \{0, 1\}^D$. Then define $\mathcal{X}_D^{\square} := \Phi[\mathfrak{C}_D^{\square}] \subset \{0, 1\}^{\mathcal{K}}$. Finally, let $\mathcal{E} := \{\mathbf{x} \in \{0, 1\}^{\mathcal{K}}; \|\mathbf{x}\| \text{ is even}\}$, and define $\mathbf{e} := \Phi[\mathcal{E}] \in \{0, 1\}^{\mathcal{K}}$.

Claim 1: For any $\mathcal{C} \in \mathfrak{C}_D^{\square}$, if $|\mathcal{C}| > 2$, and $\mathbf{c} := \Phi(\mathcal{C}) \in \mathcal{X}_D^{\square}$, then $\mathbf{c} \asymp \mathbf{e}$.

Proof: Let $\mathcal{B} \in \mathfrak{C}_D^{\square}$ and let $\mathbf{b} := \Phi(\mathcal{B}) \in \mathcal{X}_D^{\square}$. Suppose \mathbf{b} is strictly between \mathbf{c} and \mathbf{e} . Then $\mathcal{C} \cap \mathcal{E} \subseteq \mathcal{B} \cap \mathcal{E}$ and $\mathcal{C}^{\mathfrak{C}} \cap \mathcal{E}^{\mathfrak{C}} \subseteq \mathcal{B}^{\mathfrak{C}} \cap \mathcal{E}^{\mathfrak{C}}$.

Now, if $\mathcal{C} \cap \mathcal{E} \subseteq \mathcal{B} \cap \mathcal{E}$, then $\mathcal{C} \cap \mathcal{E} \subseteq \mathcal{B}$. Thus, $\text{conv}(\mathcal{C} \cap \mathcal{E}) \subseteq \mathcal{B}$ (because $\mathcal{B} \in \mathfrak{C}_D^{\square}$). But if $|\mathcal{C}| > 2$, then the definition of \mathcal{E} is such that $\text{conv}(\mathcal{C} \cap \mathcal{E}) = \mathcal{C}$. Thus, $\mathcal{C} \subseteq \mathcal{B}$. Thus, $|\mathcal{B}| > 2$ also.

Meanwhile, if $\mathcal{C}^{\mathfrak{C}} \cap \mathcal{E}^{\mathfrak{C}} \subseteq \mathcal{B}^{\mathfrak{C}} \cap \mathcal{E}^{\mathfrak{C}}$, then $\mathcal{C} \cap \mathcal{E}^{\mathfrak{C}} \supseteq \mathcal{B} \cap \mathcal{E}^{\mathfrak{C}}$. Thus $\mathcal{B} \cap \mathcal{E}^{\mathfrak{C}} \subset \mathcal{C}$. Thus, $\text{conv}(\mathcal{B} \cap \mathcal{E}^{\mathfrak{C}}) \subseteq \mathcal{C}$ (because $\mathcal{C} \in \mathfrak{C}_D^{\square}$). But if $|\mathcal{B}| > 2$, then the definition of \mathcal{E} is such that $\text{conv}(\mathcal{B} \cap \mathcal{E}^{\mathfrak{C}}) = \mathcal{B}$. Thus $\mathcal{B} \subseteq \mathcal{C}$.

We conclude that $\mathcal{B} = \mathcal{C}$, which means $\mathbf{b} = \mathbf{c}$, as desired. \diamond claim 1

Claim 2: $|\mathcal{X}_D^{\square}| = 3^D$.

Proof: $|\mathcal{X}_D^{\square}| = |\mathfrak{C}_D^{\square}|$, and $|\mathfrak{C}_D^{\square}|$ consists of all subcubes of $\{0, 1\}^D$. There is a bijective correspondence between these subcubes and the set of $\{0, 1\}$ -valued functions whose domain is any subset of $[1 \dots D]$. Any such function can be represented in a unique way by an element of $\{0, 1, *\}^D$ in the obvious way. Thus, $|\mathcal{X}_D^{\square}| = |\{0, 1, *\}^D| = 3^D$. \diamond claim 2

Now let $\mathcal{Y} := \{\mathbf{x} \in \mathcal{X}_D^{\square}; \mathbf{x} \text{ represents a subcube of cardinality 1 or 2 in } \{0, 1\}^D\}$.

Claim 3: $|\mathcal{Y}| = (1 + \frac{D}{2}) 2^D$.

Proof: Clearly, $\{0, 1\}^D$ has exactly 2^D subcubes of cardinality 1 (i.e. vertices).

To obtain a subcube of cardinality 2 (i.e. an edge), we start at one of these 2^D vertices and then move to one of its D nearest neighbours. There are $D \cdot 2^D$ ways to do this, but we have then counted every edge twice, so there are actually $\frac{D}{2} 2^D$ edges. \diamond claim 3

Thus, if D is large enough, then

$$\begin{aligned} |\mathcal{X}_D^{\square}(\mathbf{e})| &\stackrel{(*)}{\geq} |\mathcal{X}_D^{\square}| - |\mathcal{Y}| \stackrel{(\dagger)}{=} 3^D - \left(1 + \frac{D}{2}\right) 2^D \\ &= 3^D \left(1 - \left(1 + \frac{D}{2}\right) \cdot \left(\frac{2}{3}\right)^D\right) \stackrel{(\ddagger)}{\geq} \frac{1}{2} 3^D, \end{aligned}$$

where $(*)$ is by Claim 1, (\dagger) is by Claims 2 and 3, and (\ddagger) is because

$$\lim_{D \rightarrow \infty} \left(1 + \frac{D}{2}\right) \left(\frac{2}{3}\right)^D = 0.$$

Thus,

$$(A5) \quad \log_2 |\mathcal{X}_D^{\square}(\mathbf{e})| \geq \log_2 \left(\frac{1}{2} 3^D\right) = D \cdot \log_2(3) - 1.$$

Now, $\mathbf{e} \in \text{Maj}(\mathcal{X}_D^\square)$ because \mathcal{X}_D^\square is McGarvey by Lemma A.2. Thus

$$h(\mathcal{X}_D^\square) \geq \frac{\log_2 |\mathcal{X}_D^\square(\mathbf{e})|}{\log_2 |\mathcal{X}_D^\square|} \underset{(*)}{\geq} \frac{D \cdot \log_2(3) - 1}{\log_2 |3^D|} = \frac{D \cdot \log_2(3) - 1}{D \cdot \log_2(3)} \xrightarrow{D \rightarrow \infty} 1,$$

where $(*)$ is by eqn.(A5) and Claim 2. Thus, \mathcal{X}_D^\square is asymptotically fully indeterminate. \square

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DEPT. OF ECONOMICS, UNIVERSITY OF CALIFORNIA AT DAVIS, U.S.A.
E-mail address: kdnehring@ucdavis.edu

DEPT. OF MATHEMATICS, TRENT UNIVERSITY, CANADA.
E-mail address: marcuspivato@trentu.ca

Weighted voting, threshold functions, and zonotopes

Catherine Stenson

ABSTRACT. Many voting systems consist of a set of players who can form coalitions; a winning coalition is one that can pass a measure. The winning and losing coalitions can be given by several classes of functions, including switching functions, threshold functions, weighted voting systems, and simple games. We define a zonotope T_n related to these functions and describe its coordinates in terms of a measure of voting power. We also give a voting interpretation of the coordinates of the derived zonotope \bar{T}_n .

1. Introduction

In many different contexts, some collection of players (elected representatives, company shareholders, states in the U.S. Electoral College, or neurons) may form a coalition to do something (pass a bill, buy another company, choose a president, or cause another neuron to fire). Certain coalitions of players are capable of carrying out this action, and others are not. These coalitions can be described by different types of functions, including weighted voting systems, simple games, switching functions, and threshold functions. Here we describe the relationship between these functions and a geometric object known as a zonotope.

The paper is structured as follows. In this section, we define different types of voting functions. In Section 2, we describe a hyperplane arrangement \mathcal{A}_n that has been used in previous research on threshold functions. In this work, we compute the coordinates of the vertices of the zonotope T_n that is dual to \mathcal{A}_n . We show that they are determined by the critical instances used to define the Banzhaf power index. We also explain why the points of T_n that correspond to switching functions that are not threshold functions are not vertices of T_n . Given any zonotope, McMullen [8] showed how to construct its derived zonotope; in Section 3 we apply this construction to T_n to get the derived zonotope \bar{T}_n . We compute the coordinates of \bar{T}_n in two different bases and show that they also have a nice voting interpretation.

We begin with some standard definitions. Our set of players is $P = \{P_1, P_2, \dots, P_n\}$. A coalition C is a subset of P . Each coalition is either winning, meaning it can carry out an action such as passing a bill, or losing, meaning it cannot. A coalition C has a characteristic vector $\mathbf{x}_C \in \{0, 1\}^n$ with $x_i = 1$ if P_i is in C and $x_i = 0$ if it is not. We can think of \mathbf{x}_C as a vertex of the n -cube.

If each coalition C is designated as either winning or losing, we have a *simple game* provided that the set of winning coalitions satisfies *monotonicity*. That is, if

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A is a winning coalition and $A \subset B$, then B is a winning coalition. Simple games model many real-world voting systems, including multicameral legislatures. For example, let $P = \{P_1, P_2, P_3, P_4\}$ and say that a coalition is winning if and only if it contains at least one of $\{P_1, P_2\}$ and at least one of $\{P_3, P_4\}$. This is a simple game. See [13] for more about simple games.

Weighted voting systems are examples of simple games. In a weighted voting system, player P_i has v_i votes with $v_i > 0$, and a coalition C can pass a measure provided the sum of the votes of the members of C meets or exceeds the quota q . Because adding a player to a coalition does not decrease the sum of the coalition's votes, a weighted voting system is monotonic. However, not every simple game is a weighted voting system. The bicameral simple game example given above is not weighted [13, Section 1.7].

Simple games and weighted voting systems are examples of a larger class of functions called *switching functions* or *Boolean functions*. A switching function of n variables is a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$. If $f(\mathbf{x}_C) = 1$, then C is a winning coalition for f , and if $f(\mathbf{x}_C) = 0$, then C is a losing coalition for f . A switching function is a *threshold function* if there is a vector $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ and a $q \geq 0$ such that $f(\mathbf{x}) = 1$ if $\mathbf{x} \cdot \mathbf{v} \geq q$ and $f(\mathbf{x}) = 0$ if $\mathbf{x} \cdot \mathbf{v} < q$. If in addition $v_i > 0$ for $1 \leq i \leq n$, then f is a weighted voting system or *positive threshold function*. If two threshold functions f and g are given by the vectors \mathbf{v} and \mathbf{w} and quotas q_v and q_w , respectively, then f and g are *equivalent* if $f(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \{0, 1\}^n$. See [9] for more about threshold functions. The work in this paper was motivated by weighted voting systems, but it has turned out to fit more appropriately in the larger setting of threshold functions and switching functions.

There are several ways to measure the power of an individual player in a weighted voting system. The player P_i is *critical* to a coalition C if $P_i \in C$ and C is a winning coalition, but $C \setminus \{P_i\}$ is a losing coalition; this situation is a *critical instance* for P_i . Let β_i be the total number of critical instances for P_i and let $\beta = \sum_{i=1}^n \beta_i$. The (normalized) *Banzhaf power index* of a positive threshold function is $(\beta_1/\beta, \beta_2/\beta, \dots, \beta_n/\beta)$ [1]. Alternatively, the *absolute Banzhaf index* is $(\beta_1/2^{n-1}, \beta_2/2^{n-1}, \dots, \beta_n/2^{n-1})$. Equivalent ideas were developed by Penrose [11]. These critical instances are also used in the computation of the Shapley-Shubik power index [12]. For discussions of the appropriate use of these power indices, see [3] and [7]. In this paper, we are interested in the values of the β_i , rather than the application of a particular power index.

EXAMPLE 1.1. Let $n = 4$ and consider the threshold function f given by $\mathbf{v} = (2, 1, 1, 1)$ and $q = 3$. All coalitions with at least three players win, and the three 2-player coalitions involving P_1 win. No player is critical to the coalition $\{P_1, P_2, P_3, P_4\}$. Player P_1 is critical to each of the 3-player and 2-player coalitions it is in, so $\beta_1 = 6$. Player P_2 is critical to $\{P_2, P_3, P_4\}$ and to $\{P_1, P_2\}$, so $\beta_2 = 2$. Similarly, $\beta_3 = 2$ and $\beta_4 = 2$.

2. Hyperplane arrangements and zonotopes

Threshold functions have a particularly nice interpretation in terms of two geometric objects, a hyperplane arrangement and a zonotope. Here we give some of the basic definitions and properties of these objects; for further information, see [5], [8], [16], and [17].

A *hyperplane* in \mathbb{R}^d is an affine subspace of dimension $d - 1$. A *hyperplane arrangement* in \mathbb{R}^d is a collection of hyperplanes. The *regions* of a hyperplane arrangement \mathcal{A} are the connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{A}} H$, the complement of the union of the hyperplanes.

To study threshold functions of n variables, we consider the hyperplane arrangement \mathcal{A}_n in \mathbb{R}^{n+1} with 2^n hyperplanes, one for each coalition $C \subseteq \{P_1, P_2, \dots, P_n\}$, defined as follows. The hyperplane H_C is the set of all points \mathbf{v} satisfying $\mathbf{z}_C \cdot \mathbf{v} = 0$, where $\mathbf{z}_C = (-1, \mathbf{x}_C)$. We note that \mathbf{z}_C is normal to H_C .

We write a point in \mathbb{R}^{n+1} as $\mathbf{v} = (v_0, v_1, v_2, \dots, v_n)$, thinking of v_i as the number of votes of player P_i for $i \geq 1$ and v_0 as the quota. A point \mathbf{v} satisfying $\mathbf{z}_C \cdot \mathbf{v} > 0$ is on the *positive side* of H_C , and a point \mathbf{v} satisfying $\mathbf{z}_C \cdot \mathbf{v} < 0$ is on the *negative side* of H_C . A point \mathbf{v} determines a threshold function f on n variables as follows: For $\mathbf{x} \in \{0, 1\}^n$, let $f(\mathbf{x}) = 1$ if $\mathbf{x} \cdot (v_1, v_2, \dots, v_n) \geq v_0$ and $f(\mathbf{x}) = 0$ if $\mathbf{x} \cdot (v_1, v_2, \dots, v_n) < v_0$.

The following result is already known [15], but we include the proof because it gives insight into the structure of \mathcal{A}_n .

PROPOSITION 2.1 (Winder). *The regions of \mathcal{A}_n are in one-to-one correspondence with the equivalence classes of threshold functions of n variables.*

PROOF. We have already seen that a point $\mathbf{v} \in \mathbb{R}^{n+1}$ determines a threshold function. We need to show that if two points \mathbf{v} and \mathbf{w} are in the same region of \mathcal{A}_n and f and g are their corresponding threshold functions, then f and g are equivalent. If \mathbf{v} is on the positive side of H_C , then $\mathbf{z}_C \cdot \mathbf{v} > 0$. This means that $-v_0 + \mathbf{x}_C \cdot (v_1, v_2, \dots, v_n) > 0$ and C is a winning coalition for f . Similarly, if \mathbf{v} is on the negative side of H_C , then C is a losing coalition for f . The same argument applies to \mathbf{w} and g . Because the points \mathbf{v} and \mathbf{w} are in the same region of \mathcal{A}_n , they are on the same side of each hyperplane H_C . Therefore, the corresponding threshold functions f and g have the same winning and losing coalitions, and they are equivalent.

Conversely, a threshold function f given by a vector $(v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ and a quota q corresponds to the point $\mathbf{v} = (q, v_1, v_2, \dots, v_n) \in \mathbb{R}^{n+1}$. If \mathbf{v} is in a region of \mathcal{A}_n , then f corresponds to this region. If \mathbf{v} is in not a region of \mathcal{A}_n , then it is on at least one hyperplane H_C , and the coalition C has exactly q votes. Decreasing the quota of f by a sufficiently small positive amount ϵ does not change the winning and losing coalitions, so f is equivalent to the threshold function g given by (v_1, v_2, \dots, v_n) and quota $q - \epsilon$. The corresponding point $\mathbf{w} = (q - \epsilon, v_1, v_2, \dots, v_n) \in \mathbb{R}^{n+1}$ is in one of the regions of \mathcal{A}_n , and f corresponds to this region. \square

EXAMPLE 2.2. Figure 1 shows the hyperplane arrangement \mathcal{A}_1 for threshold functions of one variable; v_0 is the coordinate in the horizontal direction and v_1 is the coordinate in the vertical direction. The line $v_0 = 0$ is the hyperplane H_\emptyset and the line $v_1 = v_0$ is the hyperplane $H_{\{P_1\}}$. This hyperplane arrangement has four regions. In region I, $v_1 > v_0 > 0$, so $\{P_1\}$ is a winning coalition and \emptyset is a losing coalition. In region II, $v_0 > v_1$ and $v_0 > 0$, so $\{P_1\}$ and \emptyset are both losing coalitions. In region III, $v_0 > v_1$ and $v_0 < 0$, so $\{P_1\}$ is a losing coalition and \emptyset is a winning coalition. Finally, in region IV, $v_1 > v_0$ and $v_0 < 0$, so $\{P_1\}$ and \emptyset are both winning coalitions.

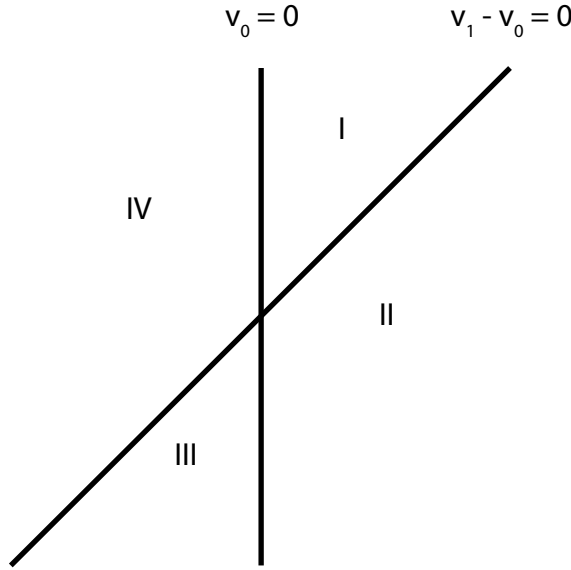


FIGURE 1. The hyperplane arrangement \mathcal{A}_1 discussed in Example 2.2.

The second geometric object we want to study is the zonotope T_n that is dual to the hyperplane arrangement \mathcal{A}_n . We begin with some preliminary definitions. Let X and Y be sets of points in \mathbb{R}^d . The *Minkowski sum* of X and Y is $X + Y = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in X \text{ and } \mathbf{y} \in Y\}$. A *zonotope* is the Minkowski sum of line segments; it is a special type of polytope. We use $[\mathbf{a}, \mathbf{b}]$ to denote the line segment between points \mathbf{a} and \mathbf{b} .

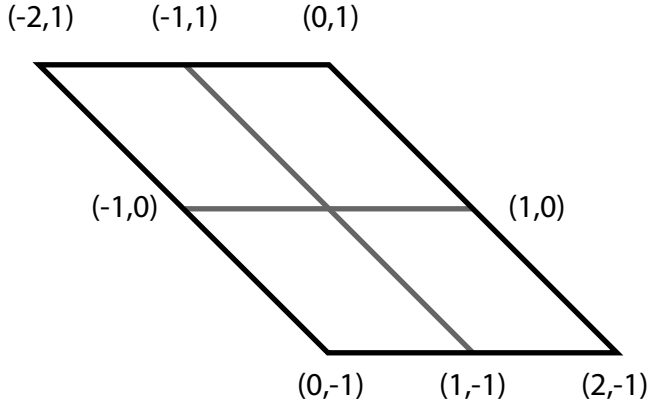
If a hyperplane arrangement has hyperplanes H_i given by $\mathbf{z}_i \cdot \mathbf{x} = 0$, $1 \leq i \leq m$, the dual zonotope is the polytope $Z = [-\mathbf{z}_1, \mathbf{z}_1] + [-\mathbf{z}_2, \mathbf{z}_2] + \cdots + [-\mathbf{z}_m, \mathbf{z}_m]$. In particular, we define

$$T_n = \sum_{C \subseteq P} [-\mathbf{z}_C, \mathbf{z}_C]$$

to be the zonotope dual to the threshold function arrangement \mathcal{A}_n .

In general, the regions of a hyperplane arrangement are in one-to-one correspondence with the vertices of the dual zonotope. (See, for example, [16, Sec. 6B] or [17, Sec. 7.3].) In particular, let $\varepsilon_i = 1$ if a region R is on the positive side of H_i and let $\varepsilon_i = -1$ if R is on the negative side of H_i . Then the region R corresponds to the vertex $\varepsilon_1 \mathbf{z}_1 + \varepsilon_2 \mathbf{z}_2 + \cdots + \varepsilon_m \mathbf{z}_m$ of Z . This correspondence between d -dimensional regions and 0-dimensional faces is part of the duality between the arrangement and the zonotope. In the case of \mathcal{A}_n , a region R is on the positive (negative) side of H_C when the coalition C is a winning (losing) coalition for the corresponding threshold function f . The corresponding vertex of T_n is $\sum_C \mathbf{z}_C - \sum_B \mathbf{z}_B$, where the first sum is over all winning coalitions C and the second sum is over all losing coalitions B . Thus, by Proposition 2.1, we have the following corollary:

COROLLARY 2.3. *The vertices of T_n are in one-to-one correspondence with threshold functions of n variables.*

FIGURE 2. The zonotope T_1 discussed in Example 2.4.

EXAMPLE 2.4. Figure 2 shows the zonotope T_1 for threshold functions of one variable, which is a parallelogram. We have $\mathbf{z}_\emptyset = (-1, 0)$ and $\mathbf{z}_{\{P_1\}} = (-1, 1)$. The line segments $[-\mathbf{z}_\emptyset, \mathbf{z}_\emptyset]$ and $[-\mathbf{z}_{\{P_1\}}, \mathbf{z}_{\{P_1\}}]$ are shown in grey. The vertex $(0, 1) = -(-1, 0) + (-1, 1)$ corresponds to the threshold functions in region I of \mathcal{A}_1 in Figure 1. Similarly, the vertices $(2, -1)$, $(0, -1)$, and $(-2, 1)$ correspond to the threshold functions in regions II, III, and IV, respectively.

We see that threshold functions correspond to points in T_n , and in fact so do all switching functions. Let f be a switching function. We define $h_f(C) = 2f(\mathbf{x}_C) - 1$, where C is a coalition, so that $h_f(C) = 1$ when C is a winning coalition for f and $h_f(C) = -1$ when C is a losing coalition for f . A switching function f corresponds to the point $\mathbf{y}_f = \sum_C h_f(C)\mathbf{z}_C \in T_n$. By the correspondence that gives Corollary 2.3, if f is a threshold function then \mathbf{y}_f is a vertex of T_n , and if \mathbf{u} is a vertex of T_n , then there is a threshold function f such that $\mathbf{u} = \mathbf{y}_f$. At the end of this section, we will see why it is not possible to also have $\mathbf{u} = \mathbf{y}_g$, where \mathbf{u} is a vertex of T_n and g is a switching function that is not a threshold function.

EXAMPLE 2.5. The function f defined in Example 1.1 has $\mathbf{y}_f = (0, 6, 2, 2, 2)$. This looks strikingly similar to our previous computation of critical instances for this threshold function.

The coordinates of \mathbf{y}_f do have a nice interpretation in terms of a signed generalization of critical instances. If $P_i \in C$ and C is a winning coalition, but $C \setminus \{P_i\}$ is a losing coalition, this is a *positive critical instance* for P_i . This is simply the usual definition of critical instance for the Banzhaf power index. If $P_i \in C$ and C is a losing coalition, but $C \setminus \{P_i\}$ is a winning coalition, this is a *negative critical instance* for P_i . Simple games have no negative critical instances. Let γ_i be the number of positive critical instances for P_i minus the number of negative critical instances for P_i . The *vector of signed critical instances* for f is $(\gamma_1, \gamma_2, \dots, \gamma_n)$.

PROPOSITION 2.6. *If $\mathbf{y}_f = \sum_C h_f(C)\mathbf{z}_C = (y_0, y_1, y_2, \dots, y_n)$ is a point of T_n corresponding to a switching function f , then the vector of signed critical instances for f is $(y_1 + \frac{y_0}{2}, y_2 + \frac{y_0}{2}, \dots, y_n + \frac{y_0}{2})$.*

PROOF. We have

$$y_i + \frac{y_0}{2} = \sum_{P_i \in C} h_f(C) - \frac{1}{2} \sum_C h_f(C).$$

We can rewrite this sum in terms of coalitions that do not contain P_i :

$$\begin{aligned} y_i + \frac{y_0}{2} &= \sum_{\substack{B \\ P_i \notin B}} h_f(B \cup \{P_i\}) - \frac{1}{2}(h_f(B \cup \{P_i\}) + h_f(B)) \\ &= \sum_{\substack{B \\ P_i \notin B}} \frac{1}{2}(h_f(B \cup \{P_i\}) - h_f(B)). \end{aligned}$$

If B and $B \cup \{P_i\}$ either both win or both lose, $h_f(B \cup \{P_i\}) - h_f(B) = 0$. If B loses and $B \cup \{P_i\}$ wins, then this is a positive critical instance for P_i and $(1/2)(h_f(B \cup \{P_i\}) - h_f(B)) = 1$. If B wins and $B \cup \{P_i\}$ loses, then this is a negative critical instance for P_i and $(1/2)(h_f(B \cup \{P_i\}) - h_f(B)) = -1$. \square

In the special case that the number of winning coalitions of f is equal to the number of losing coalitions of f , as in Example 1.1, $y_0 = 0$ and $y_i = \gamma_i$ for $1 \leq i \leq n$. In particular, this happens with functions for which the complement of a winning coalition is always a losing coalition and vice versa. For threshold functions with $q > \sum_i v_i/2$, it is possible to have a pair of complementary coalitions that both lose, and $y_0/2$ counts these pairs. Similarly, if $q \leq \sum_i v_i/2$, it is possible to have a pair of complementary coalitions that both win, and $|y_0/2|$ counts these pairs.

Now we return to considering switching functions that are not also threshold functions; for brevity, we will call them *non-threshold switching functions*. We have seen in Corollary 2.3 that threshold functions correspond to vertices of T_n , and we are about to see that non-threshold switching functions correspond to points of T_n that are not vertices. We give two proofs of the following corollary of Proposition 2.1, one that is short and standard and one that is new and gives additional insight into the geometry.

COROLLARY 2.7. *If g is a switching function that is not a threshold function, then \mathbf{y}_g is not a vertex of T_n .*

PROOF 1. This corollary is an immediate consequence the following standard result about zonotopes: If $\mathbf{w} = \varepsilon_1 \mathbf{z}_1 + \varepsilon_2 \mathbf{z}_2 + \cdots + \varepsilon_m \mathbf{z}_m$, where $\varepsilon_i = \pm 1$, then \mathbf{w} is a vertex of Z if and only if there is some region of the corresponding hyperplane arrangement that is on the positive side of H_i when $\varepsilon_i = 1$ and on the negative side of H_i when $\varepsilon_i = -1$. (A more general version of this statement is [2, Proposition 2.2.2].) In the case of T_n , this result plus Proposition 2.1 proves the corollary. \square

Corollary 2.7 raises some questions not addressed by this first proof. Because T_n is a zonotope, if \mathbf{y}_g is a point of T_n but not a vertex of T_n , we should be able to write \mathbf{y}_g as a sum of points from the line segments $[-\mathbf{z}_C, \mathbf{z}_C]$ so that for at least one C , the point is from the interior of the line segment. How do we do this? In addition, what property of non-threshold switching functions guarantees that they do not correspond to vertices of T_n ? The goal of the following proof is to answer these questions.

PROOF 2. Before we consider non-threshold switching functions, we need to know more about threshold functions. If a threshold function f is given by a vector \mathbf{v} and a quota q , the characteristic vectors in $W = \{\mathbf{x}_C : C \text{ is a winning coalition for } f\}$ are on the positive side of the hyperplane $\mathbf{x} \cdot \mathbf{v} = q - \epsilon$, where $\epsilon > 0$ is small, and the characteristic vectors in $L = \{\mathbf{x}_C : C \text{ is a losing coalition for } f\}$ are on the negative side. Conversely, if f is a switching function and a hyperplane $\mathbf{x} \cdot \mathbf{v} = q - \epsilon$ separates the characteristic vectors of the winning coalitions of f from the losing coalitions of f , then f is a threshold function. (See [13, Lemma 2.65] and [9, Section 3.3.1].)

Thus if g is a non-threshold switching function, it is not possible to separate W from L with a hyperplane. This means that the convex hull of W intersects the convex hull of L . (For standard definitions and properties of convex sets, see [5].) Thus there exist winning coalitions A_1, A_2, \dots, A_p and losing coalitions B_1, B_2, \dots, B_q such that

$$\sum_{i=1}^p \lambda_i \mathbf{x}_{A_i} = \sum_{j=1}^q \mu_j \mathbf{x}_{B_j}$$

where

$$\sum_{i=1}^p \lambda_i = 1 = \sum_{j=1}^q \mu_j \text{ and } \lambda_i > 0 \text{ and } \mu_j > 0 \text{ for all } i \text{ and } j.$$

Because $\mathbf{z}_C = (-1, \mathbf{x}_C)$ for any coalition C , we also have

$$\sum_{i=1}^p \lambda_i \mathbf{z}_{A_i} = \sum_{j=1}^q \mu_j \mathbf{z}_{B_j}.$$

Thus we can write the point \mathbf{y}_g in T_n corresponding to g as

$$\begin{aligned} \mathbf{y}_g &= \sum_{i=1}^p \mathbf{z}_{A_i} + \sum_{j=1}^q -\mathbf{z}_{B_j} + \sum_{\substack{C \\ C \neq A_i, B_j}} h_g(C) \mathbf{z}_C \\ &= \sum_{i=1}^p (\lambda_i + (1 - \lambda_i)) \mathbf{z}_{A_i} + \sum_{j=1}^q -\mathbf{z}_{B_j} + \sum_{\substack{C \\ C \neq A_i, B_j}} h_g(C) \mathbf{z}_C \\ &= \sum_{i=1}^p (1 - \lambda_i) \mathbf{z}_{A_i} + \sum_{j=1}^q (-1 + \mu_j) \mathbf{z}_{B_j} + \sum_{\substack{C \\ C \neq A_i, B_j}} h_g(C) \mathbf{z}_C. \end{aligned}$$

We note that $0 \leq 1 - \lambda_i < 1$ and $-1 < -1 + \mu_j \leq 0$. Thus $(1 - \lambda_i) \mathbf{z}_{A_i}$ is in the interior of the line segment $[-\mathbf{z}_{A_i}, \mathbf{z}_{A_i}]$ and $(-1 + \mu_j) \mathbf{z}_{B_j}$ is in the interior of the line segment $[-\mathbf{z}_{B_j}, \mathbf{z}_{B_j}]$. Thus \mathbf{y}_g is not a vertex of $T_n = \sum_{C \subseteq P} [-\mathbf{z}_C, \mathbf{z}_C]$. \square

3. The derived zonotope

Given a zonotope Z , McMullen [8] defined a second zonotope \bar{Z} . Just as the coordinates of T_n have a voting interpretation, the coordinates of \bar{T}_n have a voting interpretation.

Let $Z = [-\mathbf{z}_1, \mathbf{z}_1] + [-\mathbf{z}_2, \mathbf{z}_2] + \cdots + [-\mathbf{z}_m, \mathbf{z}_m]$ such that $\mathbf{z}_i \in \mathbb{R}^d$ and the linear span of $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m\}$ is \mathbb{R}^d . Then Z is a d -dimensional zonotope. If $m > d$, the set $\{\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_m\}$ is linearly dependent and the space of linear dependences has dimension $m - d$. Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{m-d}\}$ be a basis for this space, where $\mathbf{a}_j = (a_{j,1}, a_{j,2}, \dots, a_{j,m})$ and

$$a_{j,1}\mathbf{z}_1 + a_{j,2}\mathbf{z}_2 + \cdots + a_{j,m}\mathbf{z}_m = 0.$$

We define $\bar{\mathbf{z}}_i = (a_{1,i}, a_{2,i}, \dots, a_{(m-d),i})$. The *derived zonotope* of Z is the $(m - d)$ -dimensional zonotope $\bar{Z} = [-\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_1] + [-\bar{\mathbf{z}}_2, \bar{\mathbf{z}}_2] + \cdots + [-\bar{\mathbf{z}}_m, \bar{\mathbf{z}}_m]$. (For a discussion of \bar{Z} in terms of oriented matroid duality, see [2].) For T_n , $d = n + 1$ and $m = 2^n$.

EXAMPLE 3.1. We consider the zonotope T_2 with $\mathbf{z}_1 = \mathbf{z}_0 = (-1, 0, 0)$, $\mathbf{z}_2 = \mathbf{z}_{\{P_1\}} = (-1, 1, 0)$, $\mathbf{z}_3 = \mathbf{z}_{\{P_2\}} = (-1, 0, 1)$, and $\mathbf{z}_4 = \mathbf{z}_P = (-1, 1, 1)$. The space of linear dependences has dimension $2^2 - (2 + 1) = 1$, and $\mathbf{a}_1 = (1, -1, -1, 1)$ is a basis for it. We have $\bar{\mathbf{z}}_1 = 1, \bar{\mathbf{z}}_2 = -1, \bar{\mathbf{z}}_3 = -1$, and $\bar{\mathbf{z}}_4 = 1$, where we omit the parentheses around $\bar{\mathbf{z}}_i$ because it is a point in 1-dimensional space. The 1-dimensional zonotope $\bar{T}_2 = [-1, 1] + [-1, 1] + [-1, 1] + [-1, 1]$ is the line segment $[-4, 4]$.

There is a nice correspondence between certain points in Z and \bar{Z} . In the theorem below [8], a *boundary point* of Z is a point that is neither a vertex nor an interior point of Z .

THEOREM 3.2 (McMullen). *Let Z be a zonotope and let \bar{Z} be its derived zonotope. For each vector $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$, where $\varepsilon_i = \pm 1$, the point $\varepsilon_1\mathbf{z}_1 + \varepsilon_2\mathbf{z}_2 + \cdots + \varepsilon_m\mathbf{z}_m$ is a vertex, boundary point, or interior point of Z if and only if $\varepsilon_1\bar{\mathbf{z}}_1 + \varepsilon_2\bar{\mathbf{z}}_2 + \cdots + \varepsilon_m\bar{\mathbf{z}}_m$ is an interior point, boundary point, or vertex of \bar{Z} , respectively.*

EXAMPLE 3.3. There are 14 threshold functions of 2 variables, so T_2 has 14 vertices. For example, the threshold function for which a coalition is winning if and only if it contains P_1 gives the vertex $-(-1, 0, 0) + (-1, 1, 0) - (-1, 0, 1) + (-1, 1, 1) = (0, 2, 0)$. This vertex of T_2 corresponds to the interior point $-1 + (-1) - (-1) + 1 = 0$ of \bar{T}_2 . In contrast, the switching function with winning coalitions \emptyset and P is not a threshold function and gives the interior point $(-1, 0, 0) - (-1, 1, 0) - (-1, 0, 1) + (-1, 1, 1) = (0, 0, 0)$ of T_2 . This interior point of T_2 corresponds to the vertex $1 - (-1) - (-1) + 1 = 4$ of \bar{T}_2 . We note that this is not a one-to-one correspondence. The switching function with winning coalitions $\{P_1\}$ and $\{P_2\}$ is also not a threshold function and gives the interior point $-(-1, 0, 0) + (-1, 1, 0) + (-1, 0, 1) - (-1, 1, 1)$ of T_2 , which is also $(0, 0, 0)$. Here the corresponding point of \bar{T}_2 is the vertex $-1 + (-1) + (-1) - 1 = -4$.

Any basis of the space of linear dependences of the points \mathbf{z}_C has $2^n - n - 1$ elements. We let the basis elements correspond to the coalitions of P which have at least two members and let the 2^n entries in each basis element correspond to the coalitions of P . Different choices of basis do not affect the combinatorial structure of \bar{T}_n , but they do give different coordinates for the point $\bar{\mathbf{y}}_f = \sum_C h_f(C)\bar{\mathbf{z}}_C$, and these coordinates reveal different information about the switching function f . We describe two different ways to choose the basis.

Basis 1. Say a coalition B has $k \geq 2$ members. We define the coordinate of \mathbf{a}_B in the position corresponding to a coalition C to be

$$(3.1) \quad \mathbf{a}_{B,C} = \begin{cases} -(k-1) & \text{if } C = \emptyset \\ 1 & \text{if } C = \{P_i\} \text{ and } P_i \in B \\ -1 & \text{if } C = B \\ 0 & \text{otherwise.} \end{cases}$$

The coordinates of $\bar{\mathbf{z}}_C$ are then $\mathbf{a}_{B,C}$, where B runs over all coalitions of P with at least two members.

PROPOSITION 3.4. *We use the basis for the space of linear dependences of the points \mathbf{z}_C as given in (3.1). Let f be a switching function. If \emptyset and all the one-player coalitions $\{P_i\}$ are losing coalitions for f , then the coordinate of the point $\bar{\mathbf{y}}_f$ in \bar{T}^n corresponding to a coalition B is 0 if B is a losing coalition for f and -2 if B is a winning coalition for f .*

PROOF. The coordinate of the point $\bar{\mathbf{y}}_f$ in \bar{T}^n corresponding to a coalition B with $|B| = k \geq 2$ is

$$-(k-1)h_f(\emptyset) + (-1)h_f(B) + \sum_{P_i \in B} h_f(\{P_i\}).$$

If \emptyset and all the $\{P_i\}$ are losing coalitions, this is $k-1-h_f(B)-k = -1-h_f(B)$, which is 0 if B is a losing coalition and -2 if B is a winning coalition. \square

The requirement that \emptyset and all the one-player coalitions are losing coalitions is a reasonable one in the context of some real-world voting systems. See, for example, [14].

EXAMPLE 3.5. Let $n = 3$. The rows of the following matrix are the basis elements \mathbf{a}_B given by (3.1), where B is the coalition to the left of the row. The columns of the matrix are the points $\bar{\mathbf{z}}_C$, where C is the coalition at the top of the column.

$$\begin{array}{c} \emptyset \quad \{P_1\} \quad \{P_2\} \quad \{P_3\} \quad \{P_1, P_2\} \quad \{P_1, P_3\} \quad \{P_2, P_3\} \quad P \\ \begin{array}{l} \{P_1, P_2\} \\ \{P_1, P_3\} \\ \{P_2, P_3\} \\ P \end{array} \end{array} \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ -2 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

For example, the first row represents the linear dependence $-(-1, 0, 0, 0) + (-1, 1, 0, 0) + (-1, 0, 1, 0) - (-1, 1, 1, 0) = (0, 0, 0, 0)$. If f is the simple game in which a coalition is winning if and only if it contains both P_1 and P_2 , then $\bar{\mathbf{y}}_f = (-2, 0, 0, -2)$. The first and fourth coordinates correspond to the winning coalitions $\{P_1, P_2\}$ and $P = \{P_1, P_2, P_3\}$ and the second and third coordinates correspond to the losing coalitions $\{P_1, P_3\}$ and $\{P_2, P_3\}$.

Basis 2. Say a coalition B has $k \geq 2$ members. We define the coordinate of \mathbf{a}_B in the position corresponding to a coalition C to be

$$(3.2) \quad \mathbf{a}_{B,C} = \begin{cases} -1 & \text{if } C = \emptyset \\ 1 & \text{if } C \subset B \text{ and } |C| = k-1 \\ -(k-1) & \text{if } C = B \\ 0 & \text{otherwise.} \end{cases}$$

The coordinates of $\bar{\mathbf{z}}_C$ are then $\mathbf{a}_{B,C}$, where B runs over all coalitions of P with at least two members.

PROPOSITION 3.6. *We use the basis for the space of linear dependences of the points \mathbf{z}_C as given in (3.2). Let f be a switching function. If \emptyset is a losing coalition for f and B is a winning coalition for f , then the coordinate of the point $\bar{\mathbf{y}}_f$ in \bar{T}^n corresponding to B is $2 - 2*$ (the number of players that are critical to B).*

PROOF. The coordinate of the point $\bar{\mathbf{y}}_f$ in \bar{T}^n corresponding to a coalition B with $|B| = k \geq 2$ is

$$-h_f(\emptyset) - (k-1)h_f(B) + \sum_{P_i \in B} h_f(B \setminus \{P_i\}).$$

If \emptyset is a losing coalition and B is a winning coalition, then this sum is

$$\begin{aligned} & -k + 2 + (\text{the number of } P_i \text{ not critical to } B) - (\text{the number of } P_i \text{ critical to } B) \\ & = -k + 2 + k - 2(\text{the number of } P_i \text{ critical to } B) \\ & = 2 - 2(\text{the number of } P_i \text{ critical to } B). \end{aligned}$$

□

EXAMPLE 3.7. Let $n = 3$. The rows of the following matrix are the basis elements \mathbf{a}_B given by (3.2), where B is the coalition to the left of the row. The columns of the matrix are the points $\bar{\mathbf{z}}_C$, where C is the coalition at the top of the column.

$$\begin{array}{c} \emptyset \quad \{P_1\} \quad \{P_2\} \quad \{P_3\} \quad \{P_1, P_2\} \quad \{P_1, P_3\} \quad \{P_2, P_3\} \quad P \\ \begin{array}{l} \{P_1, P_2\} \\ \{P_1, P_3\} \\ \{P_2, P_3\} \\ P \end{array} \begin{pmatrix} -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & -2 \end{pmatrix} \end{array}$$

For example, the fourth row represents the linear dependence $-(-1, 0, 0, 0) + (-1, 1, 1, 0) + (-1, 1, 0, 1) + (-1, 0, 1, 1) - 2(-1, 1, 1, 1) = (0, 0, 0, 0)$. If f is the switching function in which a coalition is winning if and only if it has two or more players, then $\bar{\mathbf{y}}_f = (-2, -2, -2, 2)$. The first three coordinates correspond to the two-player coalitions, in which both players are critical. The fourth coordinate corresponds to $P = \{P_1, P_2, P_3\}$, which has no critical players.

4. Conclusions and future work

We have shown that the zonotopes T_n and \bar{T}_n encode important information about switching functions. It would be nice to see what other properties of zonotopes and hyperplane arrangements might give insight into voting. The facial structure of T_n should give insight into threshold functions, and the facial structure of \bar{T}_n should give insight into switching functions that are not threshold functions. In particular, there is a lovely theorem of Zaslavsky [16] that counts the regions of a hyperplane arrangement using the lattice of intersections of the hyperplanes. Enumerating the regions of \mathcal{A}_n , and hence the vertices of T_n , would give the number of threshold functions of n variables. However, describing the intersections of the hyperplanes of \mathcal{A}_n would require a description of the affine dependences of the points of the n -cube, which is a difficult problem. The problem of enumerating

threshold functions is an old and challenging one. Results are known for $n \leq 9$ [10] and for special classes of threshold functions. A good reference for early work on the problem is [9]; two examples of more recent work are [4] and [6].

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DEPARTMENT OF MATHEMATICS, JUNIATA COLLEGE, 1700 MOORE ST., HUNTINGDON, PENNSYLVANIA 16652

E-mail address: stenson@juniata.edu

The Borda Count, the Kemeny Rule, and the Permutahedron

Karl-Dieter Crisman

ABSTRACT. When thinking about choice beyond single winners, social preference functions are natural to study; these are functions where both input and output are strict rankings of n items (or possibly ties among several such rankings). Symmetry is one mathematical way to express fairness, so it makes sense to study the symmetry of these functions carefully.

Such rankings may be viewed as a permutation of the items; since pairwise comparison is also important in voting, a natural combinatorial object for studying such functions is the permutahedron. This paper analyzes a large class of social preference functions using the representation theory of the symmetric group of the permutahedron. The main result identifies the most symmetric possible family in this class, which preserves pairwise information fully; it is the one-parameter family that connects the Borda Count and the Kemeny Rule.

1. Introduction

1.1. Why Social Preference Functions? Choice questions are typically about aggregating individual preferences into a ‘societal’ preference. For example, with n choices, A_1, A_2, \dots, A_n , any individual voter’s preference is represented as a strict transitive ranking such as $A_1 \succ A_3 \succ \dots \succ A_2$; some mathematical rule then yields an aggregate result. Different types of outcomes, whether singletons or choice functions, yield different categories of functions.

In some natural situations the actual outcome should be a ranking, or some related structure. For instance, a group could choose officers (Chair, Secretary, and Treasurer) from three candidates a nominating committee gives them. The offices might have a priority order (like for succession), but their priority status is not the only point. Even more interesting, one might have a list of factories to inspect for an internal audit. Here, the cyclic order of a visiting schedule likely is more important than which factory actually is the first one visited in the year.

In any similar case, it is reasonable to assume that the output of the function is one *or more* strict rankings, just as in a voting function the output is one or more candidates. We call such a function a *social preference function*. The most famous s.p.f. is probably the Kemeny Rule.

A special class of social preference functions has been receiving some attention in recent work, especially in the two pairs of articles [23, 29] and [34, 35]. This class,

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called *simple ranking scoring functions* (SRSFs – see section 2.1), fills exactly the same role in the class of social preference functions as the usual positional scoring rules do in the class of voting rules¹.

The point is that in many situations where choice matters we may wish to consider not just the relative rank of the candidates, but where they fit into the overall order. So one goal of this paper is to introduce an interesting generalization of several well-known procedures of value in contexts beyond single-winner elections.

1.2. Symmetry. Since individual preferences may be represented as combinatorial objects, the symmetries of such objects are often of interest. Though they do not use the same formal language, all the foundational papers (e.g., [2, 4, 10, 25]) constantly refer to such symmetry². For instance, one might think of rankings as permutations of the set of candidates $\{1, 2, \dots, n\}$, and seek information about permutations which yields information about different procedures. For a game theory example, coalitions on up/down votes (ignoring abstentions) in the United Nations Security Council are simply subsets of the power set of the set of voters.

Many ‘natural’ fairness requirements in social choice can be thought of in terms of a natural group action on such combinatorial structures. Put another way, invariance of a procedure under a group action could be considered more equitable.

Indeed, symmetry under the action of the symmetric group on n candidates $\{A_i\}_{i=1}^n$ (denoted throughout by S_n) is usually known as *neutrality*; the idea is that no candidate has an unfair advantage³. The same action on the set of voters is usually useful in a game theory context; the Security Council does *not* have this symmetry, due to the five members with veto power.

It is fruitful to study a large range of rules to see how they behave with regard to various symmetries; over time, the field of social choice has moved in this direction from a more axiomatic approach. Structural papers like [18, 19] and the social preference function papers cited above are solidly within this tradition; another goal of this paper is to add to this classification literature.

1.3. Context for this work. Within the last few years, representation theory has become a tool to reframe and powerfully extend previous classifications. Orrison and his students [7] have done so in voting theory, while work of Hernández-Lamóneda, Juárez, and Sánchez-Sánchez [12] gives similar results in cooperative game theory. These techniques are also used in work of Bargagliotti and Orrison in nonparametric statistics [3].

In these papers, representations of S_n allow generalization with fewer technical challenges, with more insight into *why* the results are true. But there is more to combinatorics than permutations, and more to fairness than the symmetric group. Pairwise comparisons between candidates have been a cornerstone of voting theory analysis since Condorcet, and one may note that a ranking of candidates is not simply a permutation, but an ordering.

¹We note that [15] and some recent preprints by Pivato and Nehring address an even *more* general group of functions.

²For instance, in [25] it is crucial that every possible set of preferences be in the domain of their functions; in [4], a major assumption is that voters’ preferences of subsets of candidates obey various (anti-)symmetric partial orders.

³Party primary systems are not neutral; over the *whole* election cycle, a candidate in an uncontested primary has (at least in principle) an advantage in winning the whole thing.

The permutahedron has the right amount of structure for analyzing social preference functions while keeping pairwise behavior in mind. Its symmetry group sheds light on the structure of neutral SRSFs. The ‘extra’ symmetry of the permutahedron corresponds precisely to the well-known concept of *reversal* symmetry (see section 2.3). So the third goal of this paper is to prove significant results about neutral SRSFs utilizing the basic representation theory of the symmetry group of the permutahedron (summarized in the Appendix).

The most important result in this classification is an explicit characterization of the most symmetric possible rules in this family – a characterization which connects the two most important members of it.

MAIN THEOREM. *If a neutral SRSF is compatible with pairwise information and fully preserves this information, then it is a rule along the one-parameter family of procedures connecting the Borda Count and the Kemeny Rule.*

‘Pairwise information’ means information about head-to-head comparisons between alternatives; see [7], Definition 4.6, and Theorem 5.12 for full details. By adding one final symmetry, one can characterize the Borda Count among SRSFs in the same way as is usually done among positional scoring rules, or rules relying only on pairwise information. Conversely, one can start moving beyond the ‘Borda versus Condorcet’ ways of thinking and start to explore *how much* of each behavior one might want in a choice procedure.

The remainder of the paper addresses the goals as follows:

- Review social choice definitions and introduce the permutahedron
- Motivate machinery with explicit statements and examples for $n = 3$
- Introduce all remaining needed concepts, and prove theorems for all n
- Look forward to questions opened up by this work, including other discrete structures of interest in social choice

2. Social Choice and Symmetry

2.1. General Definitions. We begin with relevant voting theory definitions, mostly using notation from the most relevant references ([5, 16, 17, 27, 30]).

Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a set of n candidates/alternatives; generic alternatives are given by capital letters such as A, B, C or X, Y, Z . Let $L(\mathcal{A})$ be the set of (strict) linear rankings of those alternatives, such as $A_1 \succ A_2 \succ \dots \succ A_n$. Rankings correspond to permutations of the n elements of \mathcal{A} , and we will often identify $L(\mathcal{A})$ and S_n by abuse of notation; it should always be clear which is intended. Given a ranking r , if $X \succ Y$ in the order implied by r , we say that $X \succ_r Y$. Likewise, for any $1 \leq i \leq n$ the i th ranked alternative in r is denoted $r(i)$.

A *profile* \mathbf{p} is a vector-valued function $\mathbf{p} : L(\mathcal{A}) \rightarrow \mathbb{Q}$, where one interprets each value as the number of voters⁴ who prefer a given ranking in $L(\mathcal{A})$. Under this interpretation, the notation $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)f(v)$ signifies evaluating some function f over each ranking $v \in L(\mathcal{A})$ with multiplicity $\mathbf{p}(v)$, the ‘number of voters preferring v in \mathbf{p} ’. As an example, let $f(v) = 1$ if $v(1) = A$ and 0 otherwise; then this sum simply counts the number of voters putting A in first place.

A *social preference function* is a function from the set of all (finite) profiles to the power set of $L(\mathcal{A})$ (excepting the empty set). One might think of a social

⁴Although applications may have only integer numbers of voters, it is common practice to use \mathbb{Q} to have a vector space, which also enables normalization, if desired.

preference function as taking an electorate's set of preferences and yielding some nonempty set of rankings. The simplest social preference functions are obtained by taking a social welfare function or voting rule \mathcal{F} , and then returning the set of all full rankings which do not disagree with the outcome of \mathcal{F} .

DEFINITION 2.1. Let a *weighting vector* be an arbitrary⁵ $\mathbf{w} \in \mathbb{R}^n$. Given a profile \mathbf{p} , we say an alternative X receives the *score* $\sum_{i=1}^n \sum_{v \in L(\mathcal{A}), v(i)=X} \mathbf{p}(v)\mathbf{w}(i)$ from the weighting vector; the set of alternatives with the maximum such score is the set of *winners*. The *positional scoring rule* associated to w is the social preference function which associates to each profile \mathbf{p} all rankings r in which all winners are ranked above all non-winners.

Even viewed as preference functions, positional scoring rules are familiar. The vector $\mathbf{w} = (1, 0, 0, \dots, 0)$ gives the plurality vote, and $\mathbf{w} = (n-1, n-2, \dots, 1, 0)$ yields the *Borda Count* (BC).

2.2. Neutral Simple Ranking Scoring Functions. Our main objects of study are functions which essentially give scores to full rankings rather than individual candidates. The following definition is due to Conitzer et al. [5], though it is very similar to a roughly contemporaneous definition of generalized scoring rules in Zwicker [35] (in the case $I = O = L(\mathcal{A})$)⁶.

DEFINITION 2.2. A social preference function f is a *simple ranking scoring function* (SRSF) if there exists a function $s : L(\mathcal{A}) \times L(\mathcal{A}) \rightarrow \mathbb{R}$ such that for all votes v , $f(v)$ is the ranking(s) r which maximizes $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$. The function f is *neutral* if s is neutral; that is, if for any $\sigma \in S_{\mathcal{A}} = S_n$, $s(v, r) = s(\sigma(v), \sigma(r))$.

If an SRSF is neutral, then $s(v, r) = s(r, v)$, because we could have $\sigma = \pi^*$, where $\pi^*(v) = r$ and vice versa. For example, one could have s defined so that $s(v, r) = 1$ if $r = v$ and $s(v, r) = 0$ otherwise; in this case, we have the s.p.f. analogous to plurality, where the most popular ranking (or rankings) in the profile is the winning one.

The neutral SRSF concept is powerful, as it generalizes two otherwise disparate systems. One of these is the following often-studied (though less often used in practice) rule, as in Proposition 2 of [5].

DEFINITION 2.3. Let v and r be rankings, and $A, B \in \mathcal{A}$; then the following function measures agreement between v and r on the candidates A and B :

$$\delta(v, r, A, B) = \begin{cases} 1, & A \succ_r B \text{ and } A \succ_v B \\ 0, & \text{otherwise} \end{cases}.$$

The *Kemeny Rule* (KR) is the neutral SRSF with $s(v, r) = \sum_{A, B \in \mathcal{A}} \delta(v, r, A, B)$.

This definition is notationally dense (see [14] for the original, in terms of metrics). One should interpret this as saying that the Kemeny Rule evaluates a vote for the ranking v by assigning $\binom{n}{2}$ points to the ranking v , $\binom{n}{2} - 1$ points to any ranking r differing by one switch of places from v (i.e. switching $v(i)$ and $v(i+1)$ for some i), and so on, down to no points to the ranking which reverses v completely.

⁵Usually one requires the entries to be nonincreasing as a function of index, but a priori this need not be so.

⁶Conitzer introduces these to study 'maximum likelihood estimators', while Zwicker puts them in a more general (and geometric) context.

Then one adds up points as usual to determine the ‘winning’ ranking(s). KR is considered to be particularly important because it is the unique preference function which is a neutral and consistent Condorcet extension (see [30]).

EXAMPLE 2.4. Let’s use the KR for a profile \mathbf{p} for $n = 3$ with $\mathbf{p}(ABC) = 4$ (four voters choose ABC), $\mathbf{p}(BCA) = 3$, and no one chooses any other rankings ($\mathbf{p}(r) = 0$ for other rankings r). The maximum number of votes for a ranking is $\binom{n}{2} = 3$. In Figure 1 we can see how far distant each ranking is from the two with actual votes (for example, ACB is adjacent to ABC and CAB , even though the latter would seem far away in a single-winner context).

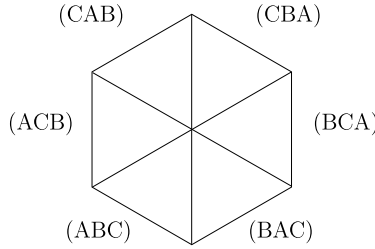


FIGURE 1

Put it together, and the winning ranking under KR is ABC :

ABC	$4 \cdot 3 + 3 \cdot 1 = 15$	CBA	$4 \cdot 0 + 3 \cdot 2 = 6$
BAC	$4 \cdot 2 + 3 \cdot 2 = 14$	CAB	$4 \cdot 1 + 3 \cdot 1 = 7$
BCA	$4 \cdot 1 + 3 \cdot 3 = 13$	ACB	$4 \cdot 2 + 3 \cdot 0 = 8$

On the other hand, every positional scoring rule is a neutral SRSF as well (Proposition 1 of [5]). Given a vote v and a candidate A , let $t(v, A)$ be the number of points that A gets if someone votes v . Then if we denote the i th-place candidate in a ranking r by $r(i)$, the function

$$s(v, r) = \sum_{i=1}^n (n - i)t(v, r(i))$$

turns a positional scoring rule into an SRSF. Intuitively, the SRSF score for v with respect to a ranking r is the sum of points each candidate in ranking r gets for vote v in the scoring rule, weighted by the position of the candidate in the ranking r .

We can make all this quite concrete with three candidates. For a positional scoring rule, if $v = XYZ$ and $r = ABC$, then

$$\begin{aligned} s(v, r) &= \sum_{i=1}^3 (3 - i)t(XYZ, r(i)) \\ &= 2 \cdot t(XYZ, A) + 1 \cdot t(XYZ, B) + 0 \cdot t(XYZ, C) \\ &= 2t(XYZ, A) + t(XYZ, B), \end{aligned}$$

so if we have a system with $\mathbf{w} = (u, w, 0)$, then this yields

v	ABC	ACB	CAB	CBA	BCA	BAC
$s(v, ABC)$	$2u + w$	$2u$	$2w$	w	u	$2w + u$

Figure 2 gives a visualization using a Saari-like triangle, where each separating line defines the border between rankings with $X \succ Y$ and vice versa. To use it for other $v' \neq v$, one would permute the whole triangle with the permutation σ such that $\sigma(v) = v'$. Then computing the SRSF for each $r = XYZ$ can be done visually as well, by taking the dot product of the profile and the weighting triangle with ABC on the XYZ spot.

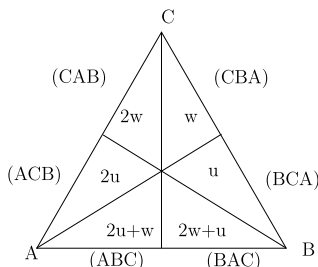


FIGURE 2. Visualizing positional scoring rules as SRSFs

EXAMPLE 2.5. It is instructive to see what plurality looks like as an s.p.f. Since $t(v, r(i)) = 0$ unless $r(i) = v(1)$, in which case we get $s(v, r) = n - i$, the score for r is $\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$, which is the sum of $n - 1$ points for each voter who ranks $r(1)$ first, $n - 2$ points for each one who ranks $r(2)$ first, and so forth.

For $n = 3$, with the profile from Example 2.4, we see that ABC is again the aggregate preference.

r	$\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$
ACB	$4 \cdot 2 + 3 \cdot 0 = 8$
ABC	$4 \cdot 2 + 3 \cdot 1 = 11$
BAC	$4 \cdot 1 + 3 \cdot 2 = 10$
BCA	$4 \cdot 1 + 3 \cdot 0 = 4$

EXAMPLE 2.6. On the other hand, the Borda Count gives BAC as the winning ranking. Putting $u = 2$ and $w = 1$ gives the following SRSF-style scores.

r	$\sum_{v \in L(\mathcal{A})} \mathbf{p}(v)s(v, r)$
ACB	$4 \cdot 4 + 3 \cdot 1 = 19$
ABC	$4 \cdot 5 + 3 \cdot 2 = 26$
BAC	$4 \cdot 4 + 3 \cdot 4 = 28$
BCA	$4 \cdot 2 + 3 \cdot 5 = 23$

Just as the analysis of [7, 18, 19] considers the point totals to be vital information in understanding voting function symmetry, we consider the point totals for SRSFs to be vital to unlocking the structure of social preference functions.

2.3. The Permutahedron and Reversal Symmetry. The input profiles and output scores of SRSFs are both essentially elements of an $n!$ -dimensional vector space over \mathbb{Q} . Since we identify rankings with permutations, we identify this space with the group ring $\mathbb{Q}S_n$, which is the set of all formal \mathbb{Q} -sums $\sum_{\sigma \in S_n} q_\sigma \sigma$.

To be specific, we are equating σ with the ranking r such that $r(\sigma(i)) = X_i$.⁷ In addition, *neutral* SRSFs have *all* their $s(v, r)$ given by *one* vector in this same space, since $s(\sigma(v), r) = s(v, \sigma^{-1}(r))$. Hence, to analyze neutral SRSFs, we will want to look at this structure. See Section 3 for concrete examples when $n = 3$.

It is time to introduce the other major player in our story. Traditional analyses of both positional scoring rules and the Kemeny rule involve yet another symmetry – the concept that if everyone reverses *all* their preferences, then the final outcome should be reversed as well.

Let $\rho = (1, n)(2, n - 1) \cdots$ be the so-called ‘reversal’ element of S_n . Then the ranking corresponding to $\rho\sigma$ is the one such that $r(\rho\sigma(i)) = r(n + 1 - \sigma(i)) = X_i$, or in other words the strict reversal of the ranking corresponding to σ (like $A \succ B \succ C$ is the reversal of $C \succ B \succ A$). For a general ranking v , we denote its reversal by v^ρ ; we will use the same notation for the operation of reversing all rankings in a set or changing $\mathbf{p}(v)$ to $\mathbf{p}(v^\rho)$ for all v in a profile.

DEFINITION 2.7. We say a social preference function f has *reversal symmetry* if $[f(\mathbf{p})]^\rho = f(\mathbf{p}^\rho)$ for all profiles \mathbf{p} .

Not all SRSFs observe this symmetry, not even simple ones like plurality. Any profile \mathbf{p} with 25% each preferring $A \succ B \succ C$, $A \succ C \succ B$, $C \succ B \succ A$, and $B \succ C \succ A$ suffices, as with plurality the winning rankings are ABC and ACB whether one uses \mathbf{p} or \mathbf{p}^ρ . Examples of rules which *do* have reversal symmetry are BC and KR. They are symmetric with respect to the following combinatorial object.

DEFINITION 2.8. The n -*permutahedron* Π_n is the graph with $n!$ vertices, indexed by permutations of the set $\{1, 2, \dots, n\}$ (or elements of S_n , as preferred), and with an edge connecting permutations σ and σ' if and only if $\sigma' = (i, i + 1)\sigma$ for some $1 \leq i < n$. (See [32] for more details.) We call its symmetry group P_n .

Another way to say this same definition is that the permutahedron is the Cayley graph of the symmetric group S_n for the neighbor-swap generating set

$$\{(1, 2), (2, 3), \dots, (n - 1, n)\} .$$

Since ρ simply changes i to $n + 1 - i$ in a permutation, a neighbor-swap $(i, i + 1)$ will become $(n + 1 - i, n + i)$, so all edges are preserved by ρ , which means this graph is just the tool for looking at reversal symmetry.

The 3-permutahedron is in fact the (graph associated to the) regular hexagon, where the vertices are labeled by permutations of $\{1, 2, 3\}$, written as reduced words. Just in the case of dimension three this also works by labeling edges instead, so we do this in Figure 3 because of the useful analogies with the representation triangle and Figure 1. This helps visualize neutral SRSFs which satisfy reversal symmetry. Think of the score for a ranking r as being a sort of ‘dot product’ of the profile with the hexagon, except that rankings the same distance from r get the same score.

For example, Figure 4 gives the vector of weights and the profile for Example 2.4. Imagine rotating the XYZ hexagon so that XYZ is on each ranking on the right; the sum of the products of each region will be the same as before.

Figure 5 shows the 4-permutahedron (where vertices are again labeled).

⁷For instance, $\sigma = (123)$ will correspond to $r(1) = r(\sigma(3)) = X_3$, $r(2) = r(\sigma(1)) = X_1$, and $r(3) = r(\sigma(2)) = X_2$, or $X_3 \succ X_1 \succ X_2$.

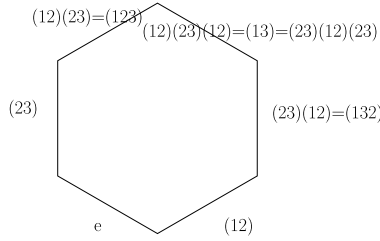


FIGURE 3. The 3-permutahedron

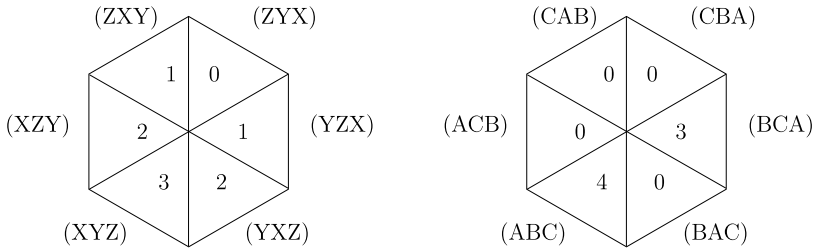


FIGURE 4. Visualizing the scoring of the Kemeny Rule

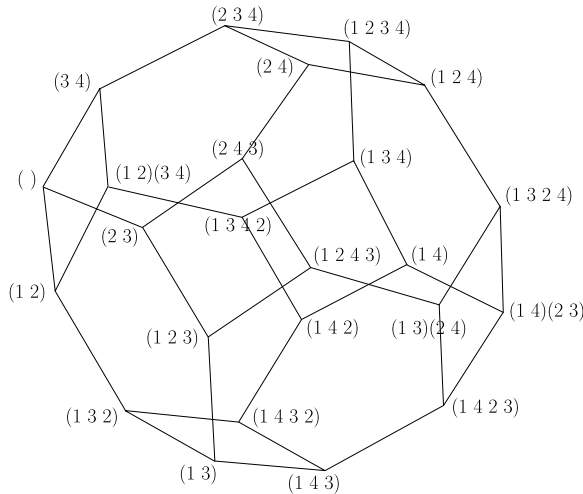


FIGURE 5. The 4-permutahedron

Reversal symmetry is important, but to ensure no symmetries are missed, one needs the overall symmetry group – that is, what is P_n ? The answer is more mathematical folklore⁸ than otherwise, but it turns out that $P_n \cong S_n \times C_2$, where C_n is the cyclic group of order n . The C_2 subgroup is precisely⁹ that given by ρ , the reversal symmetry!

⁸See [6, 9, 31] for proofs and discussion.

⁹Since it has index two, it is clear that $S_n \triangleleft P_n$. In particular, if one thinks of S_n as acting by right multiplication on the permutahedron, this provides a natural inclusion inside P_n . Then, since we already saw that ρ acts on the left on the permutahedron, the action given by reversal

3. Decompositions and Voting

The full power of representation theory for analyzing neutral SRSFs requires enough machinery that we postpone it to Section 4. In this section, we use this power *implicitly*, and motivate our theorems with detailed examples, preliminary assertions, and informal proofs when $n = 3$.

Recall that the profile space in question is isomorphic to $\mathbb{Q}S_3$, a six-dimensional space. Importantly, the image of an SRSF *also* will be a subspace of $\mathbb{Q}S_3$. That means the decomposition of the profile space into Basic/Borda, Reversal, Condorcet, and Kernel components, first fully introduced in [17], can be given an explicit basis in $\mathbb{Q}S_3$.

K	$(1, 1, 1, 1, 1, 1)$	$S^{(3)}$
B_A	$(2, 2, 0, -2, -2, 0)$	$S_1^{(2,1)}$
B_B	$(0, -2, -2, 0, 2, 2)$	
R_A	$(1, 1, -2, 1, 1, -2)$	$S_2^{(2,1)}$
R_B	$(-2, 1, 1, -2, 1, 1)$	
C	$(1, -1, 1, -1, 1, -1)$	$S^{(1,1,1)}$

We use group elements in the order¹⁰ $e, (23), (123), (13), (132), (12)$. The representation-theoretic notation for the subspaces is in the right column; this notation makes it clear the Reversal and Borda components, whose basis elements are not orthogonal to each other, must be considered as inherently two-dimensional. (The use of B_X and C to indicate profiles should not cause ambiguity with generic candidate names.) Finally, note that the sum of the entries of each vector (except the first) is zero; such a vector is called *sum-zero*, and such profiles are called *profile differentials*, inasmuch as they do not represent actual voters.

Let f be a neutral SRSF. Recall (Subsection 2.2) that f is uniquely defined by all its $s(\cdot, r)$, which we may consider to be a vector of weights \mathbf{s} ; we will call the function $f_{\mathbf{s}}$ to indicate this fact. Thus the scores for all rankings r are simply the dot products $\sigma(\mathbf{s}) \cdot \mathbf{p}$ from before, so $f_{\mathbf{s}}$ gives a linear transformation from $\mathbb{Q}S_3$ to itself. This is given by the scores for each ranking as in the previous section¹¹.

But $f_{\mathbf{s}}$ is not just any linear transformation. Since we pointed out earlier that $s(v, r) = s(\sigma(v), \sigma(r))$ for any permutation $\sigma \in S_n$, $f_{\mathbf{s}}$ must preserve all group symmetries; so, $f_{\mathbf{s}}$ is what is called an S_3 -module homomorphism (see Section 4.1 for more detail). This means SRSFs are subject to the following basic fact:

symmetry defines exactly the subgroup C_2 which gives a direct product, since left and right multiplication will commute.

¹⁰Corresponding to the usual voting theory order $ABC, ACB, CAB, CBA, BCA, BAC$.

¹¹So that for KR we would have the matrix
$$\begin{pmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 2 & 3 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

SCHUR'S LEMMA FOR $n = 3$. The image of any of K , B , R , or C under a neutral SRSF will only be a (possibly zero) multiple of itself, except B_X and R_X may be sent to linear combinations of each other¹².

The projections of any profile onto these subspaces also are constrained by Schur's Lemma. So if we count dimensions, that means a general neutral SRSF has six degrees of freedom, which makes sense since \mathbf{s} has six independent entries.

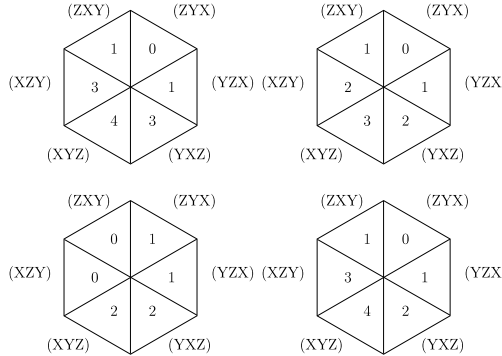


FIGURE 6. Various SRSFs – Borda, Kemeny, ‘Nonsense’, Borda Variant

EXAMPLE 3.1. Figure 6 shows a few examples of weighting vectors. The first two are the BC (rescaled) and the KR, which have already been met before.

The third appears to be an amusing nonsense procedure where $\mathbf{s} = (2, 0, 0, 1, 1, 2)$. Here, $f_{\mathbf{s}}(B_A) = -2R_A$; that is, a voter profile which overwhelmingly approves of rankings with A first ahead of other rankings would have a result overwhelmingly favoring any ranking with A in second place!

On the other hand, the last procedure (with $\mathbf{s} = (4, 3, 1, 0, 1, 2)$) is a ‘reasonable’ variant on the Borda Count which is trying to imitate plurality a little bit by deemphasizing YXZ as an outcome by voters who chose XYZ – perhaps with a view toward making sure ACB is the outcome more often with profiles like the one from Example 2.4. (It is *not* a positional scoring rule.)

Nonetheless, this \mathbf{s} has a nonzero dot product with the \mathbf{s} in the third procedure, so for some profiles with a large B_A component relative to the others, it will exhibit much the same bizarre behavior and should also be called into question.

Our main interest is in decomposing the *images* of SRSFs, but the algebra also identifies building blocks for sensible procedures. The following table gives basis vectors for \mathbf{s} which send each component (subspace) only to a scalar multiple of itself and kill everything else.

K	$(1, 1, 1, 1, 1, 1)$
C	$(1, -1, 1, -1, 1, -1)$
B	$(2, 1, -1, -2, -1, 1)$
R	$(2, -1, -1, 2, -1, -1)$

¹²The X basis vector of these components must go to a linear combination of the X vectors (or their orthogonal complements within the B and R modules). This is because these vectors have symmetry under any σ which switches alternatives Y and Z , while the others do not, and $f_{\mathbf{s}}$ is an S_3 -module homomorphism.

The other two dimensions' worth of \mathbf{s} are not reasonable – for instance, the third procedure in Example 3.1 is a basis for *any* procedure which behaves like it.

Notice that simply rescaling the Borda Count weighting vector so it is sum-zero gives the prototype for methods which preserve only the Basic component, as expected from [18]. In fact, by ignoring the part of $f_{\mathbf{s}}$ coming from K above in a systematic way (because it will always add the same amount to each ranking r), we can reduce our attention to sum-zero \mathbf{s} . Likewise, we only care about *relative* scores, which leads to the following definition (as in [7], [17], and elsewhere).

DEFINITION 3.2. We consider two procedures $f_{\mathbf{s}}$ and $f_{\mathbf{s}'}$ with sum-zero weighting vectors \mathbf{s}, \mathbf{s}' to be *equivalent* if $\mathbf{s} = k\mathbf{s}'$, which we will indicate by $s \sim s'$. We will say that two neutral SRSFs are *essentially different* if they are *not* equivalent in this way.

PROPOSITION 3.3. *The space of essentially different neutral SRSFs for $n = 3$ is four dimensional.*

INFORMAL PROOF. There are six total dimensions. Taking the quotient by K to get a sum-zero weighting vector removes one; unscaling removes another. \square

In this case, plurality has $\mathbf{s} \sim (1, 1, -1, -1, 0, 0)$. We will consider the ‘reversed’ plurality $\mathbf{s}' \sim (-1, -1, 1, 1, 0, 0)$ to be equivalent, since it will literally have a reversed outcome which can be derived from plurality – even though it looks quite different to the voter. (We are intentionally ignoring issues such as unanimity to get the broadest possible result at this point.)

Many theorists argue that the Condorcet (C profile) component should be ignored (or, what is equivalent, considered a complete tie). The idea is that any profile non-orthogonal to this component runs the risk of giving credence to subspaces where each candidate appears in each position in the ranking an equal number of times. If we do ignore it, we lose another dimension:

PROPOSITION 3.4. *The space of neutral SRSFs for $n = 3$ which ignore the Condorcet component is three dimensional.*

In the contexts mentioned in the introduction, insisting on this restriction does not always make sense. The famous Condorcet example of \mathbf{p} with $\mathbf{p}(ABC) = \mathbf{p}(BCA) = \mathbf{p}(CAB) = 1$, $\mathbf{p}(XYZ) = 0$ otherwise need not be a paradox in the committee example; there, it is plausible that the voters would prefer to have a random tiebreaker among just these three rankings (as opposed to all six), guaranteeing at least one of their succession preferences.

Naturally, not every application will demand preserving the Condorcet component, and we are not arguing that the Condorcet criterion or Condorcet extensions are always appropriate. Rather, it seems reasonable that in situations where the overall ranking matters more than the winner, or where there is potential for the ranking to influence (or even determine) a cycle of events, it is advantageous to keep this component¹³.

One of the ways in which we can ensure that we do not throw away this information is by means of the concept of being *compatible with pairwise information*. Definition 4.6 gives a full account, but for $n = 3$ it is sufficient to remark that such an $f_{\mathbf{s}}$ kills everything from R and K ; the intuition is that only B and C preserve the

¹³We briefly mention *cyclic orders* in Example 6.1.

information we get from tallying all the head-to-head pairwise matchups between candidates. By counting dimensions once again one can compute that, modulo equivalence:

PROPOSITION 3.5. *The space of neutral SRSFs for $n = 3$ which are compatible with pairwise information is two dimensional.*

Unfortunately, the procedure with $\mathbf{s} = (2, 0, 0, 1, 1, 2)$ (recall, where $f_{\mathbf{s}}(B_A) = -2R_A$) is in this space. So this is not a panacea.

Now we bring in the permutahedron, with its reversal symmetry. All of the components of the decomposition of $\mathbb{Q}S_3$ have a natural action of ρ as well (so they are “ P_3 -modules”). It is not hard to see that $(B_X)^\rho = -B_X$ while $(R_X)^\rho = R_X$, so although B and R are equivalent under S_3 , they are *not* equivalent under reversal. The implication in the social preference function context is that with reversal symmetry, B_X and R_X must not go to each other. Thus we have:

PROPOSITION 3.6. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry is two dimensional.*

INFORMAL PROOF. There are six total dimensions, and as usual we eliminate two by considering sum-zero essentially different procedures. Ordinarily, a basis element B_X of the basic subspace could be sent to some element of the reversal space, and vice versa, but if not, then quotienting out eliminates two additional dimensions. \square

The bizarre SRSF $\mathbf{s} = (2, 0, 0, 1, 1, 2)$ is not allowed, nor is anything which shares a nontrivial piece of it. However, SRSFs having reversal symmetry lead to the same problems one gets from positional scoring procedures which do not ignore the reversal component. Here is a somewhat subtle example – again, this is *not* a positional scoring procedure.

EXAMPLE 3.7. The weighting vector $\mathbf{s} = (15, 2, 0, 11, 0, 2)$ puts appropriately heavy weight on XYZ and some weight on its neighbors. Consider the profile $\mathbf{p} = (9, 6, 6, 3, 0, 6)$ with these weights; ABC (the first ranking in the profile) will be given $15 \cdot 9 + 2 \cdot 6 + 0 \cdot 6 + 11 \cdot 3 + 0 \cdot 0 + 2 \cdot 6 = 192$ points, while BAC (the last ranking in the profile) receives $15 \cdot 6 + 2 \cdot 0 + 0 \cdot 3 + 11 \cdot 6 + 0 \cdot 6 + 2 \cdot 9 = 174$.

The final score vector is $(192, 120, 174, 156, 84, 174)$; note that ACB loses by a significant margin, even to CBA , despite the pairwise tally showing A the clear victor and B tied with C .

With counterintuitive results coming no matter what restrictions we place on the symmetry, what happens if we demand maximum symmetry from a neutral SRSF?

PROPOSITION 3.8. *The space of essentially different neutral SRSFs for $n = 3$ which obey reversal symmetry and are compatible with pairs is a one-dimensional family of procedures.*

For $n = 3$, one may think of this as giving the space of $f_{\mathbf{s}}$ of the sum-zero vectors of weights in Figure 7.

Clearly both BC ($a = 2$) and KR ($a = 3$) are part of this continuum; since it is one-dimensional, they define it as well. This is our main result in the case $n = 3$.

Using a vector of weights from Figure 7 alone can still lead to a nonsense method; for example, letting $a = 0$ gives something less than useful. However, it

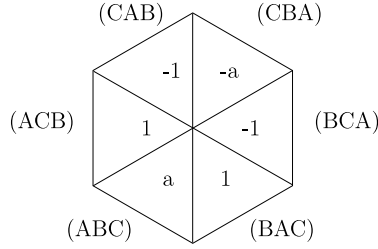


FIGURE 7. The continuum of weights for Borda-Kemeny line, $n = 3$

is very hard to decide what sort of other, *non-algebraic*, conditions are natural. One can impose unanimity-style conditions such as $a \geq 1$, under which any profile in which all voters have the same preference yields that preference as a winning ranking. Nonetheless, even then it is possible for SRSFs in this space to give actual outcomes (winning preference orders) which are different from both BC and KR.

EXAMPLE 3.9. The profile $\mathbf{p} = (1, 2, 5, 0, 0, 0)$ only has non-zero preference for half the rankings (ABC , ACB , and CAB). For a given a , this means the scores for the relevant potential winning rankings will be:

	Computation	BC	KR	$a = 1.5$
ABC	$+a \cdot 1 + 1 \cdot 2 - 1 \cdot 5$	0	-1	-1.5
ACB	$+1 \cdot 1 + a \cdot 2 + 1 \cdot 5$	12	10	9
CAB	$-1 \cdot 1 + 1 \cdot 2 + a \cdot 5$	16	11	8.5
CBA	$-a \cdot 1 - 1 \cdot 2 + 1 \cdot 5$	0	1	2

The Kemeny Rule and Borda Count both give CAB as the winning outcome, but with $a = 1.5$ the result is ACB .

However, it turns out that if $2 < a < 3$, this is not possible – all SRSFs of this type will have the same outcome as KR or BC (or both). Demonstrating this is a standard and tedious chase of inequalities to yield contradictions from other cases, so we omit the proof.

For the reader who enjoys an exercise, here is an unscaled, non-zero-sum (and hence more intuitive to the layman) example of an SRSF ‘between’ the most familiar examples.

EXAMPLE 3.10. The procedure in Figure 8 yields a tie between ABC and ACB (the BC and KR outcomes, respectively) on the profile $\mathbf{p} = 6B_A + 2B_B - 7C - 3R_C + 12K$.

Hints: which $2 < a < 3$ would this correspond to? What contribution will K make in the finally tally? What about C ? Can you now reconstruct the *relevant* part of the profile and finish the computations as in Example 3.9?

These examples show that there is real depth in the concept of simple ranking scoring functions. In order to avoid the problem in Example 3.9, we must use algebra – for instance, we could send the profile differential C to a *positive* scalar multiple of itself. To ensure that the outcome is ‘between’ those of the well-known methods (if this is even a good idea), we must come up with appropriate Pareto-type conditions.

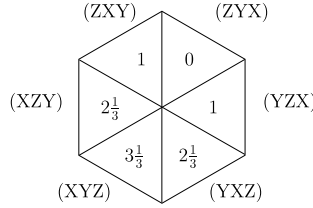


FIGURE 8. A procedure between Borda and Kemeny

It is by no means obvious how much weight to give the C component, but once one bothers with the pairwise information, its effects must be considered. If there is some dissatisfaction in the community about the Borda Count completely ignoring this information, there is also dissatisfaction with methods that give it as much weight as the Kemeny Rule does, frequently exhibiting paradoxes as the no-show paradox.

With this spectrum, each potential ‘customer’ of methods on this ‘Borda-Kemeny spectrum’ can decide this for themselves how much or how little to take this into account; the algebraic access to these procedures makes this analysis possible.

4. Representations

Let \mathcal{A} be a set of n candidates, S_n be the symmetric group on n elements, and so forth. Our results for $n \geq 3$ may be summarized as follows.

- The space of essentially different neutral SRSFs which are compatible with pairs is $\frac{1}{2}(n+1)(n-2) = \frac{1}{2}(n^2 - n + 2)$ dimensional (Theorem 5.1).
- If these also have reversal symmetry, we are reduced to $\frac{1}{4}(n^2 - 5)$ or $\frac{1}{4}(n^2 - 4)$ dimensions for odd and even n , respectively, which is about half as many (Theorem 5.4).
- Unsurprisingly, there are n dimensions of positional scoring rules, $\lfloor \frac{n}{2} \rfloor$ of which obey reversal symmetry (Theorems 5.5 and 5.6).
- An SRSF which is a scoring rule *and* is pairwise compatible is essentially the same as the Borda Count or its reversal (Corollary 5.9).
- Consider SRSFs which are pairwise compatible *and* preserve the information from pairwise matchups as fully as possible. There is just one dimension of essentially different neutral SRSFs in this family, and it is defined by the continuum connecting the Borda Count and the Kemeny Rule (Theorem 5.12).

The main importance of most of these theorems is to emphasize *how many* different SRSFs there are if we only use *some* symmetry. The number grows as $O(n^2)$ for most – the curious reader may also skip ahead to Theorem 7.1 for full, precise details and tables of the dimensions of the decompositions. Given that the number of pairwise votes that can happen grows at this rate as well, this is not surprising; nonetheless, it serves to highlight the surprising parsimony of Theorem 5.12 and Corollary 5.9.

In order to explain these theorems in full generality in Section 5, we will first give a brief review of how representation theory, modules, and voting theory can interact, then give the relevant parts of the representation theory of the permutahedron for all n .

4.1. Representation Theory, Modules, and Voting. For SRSFs, recall that we may consider both the domain and target of a given function $f_{\mathbf{s}}$ to be a vector space of dimension $n!$. Because of the role of permutations in the voting context, we will in fact consider the vector space to be the group ring $\mathbb{Q}S_n$; this has a natural S_n action by concatenating permutations. To keep notation consistent with [7], we will also often call this vector space $M^{1,1,\dots,1} = M^{1^n}$; for any partition λ of n , there is a corresponding module M^λ of profiles of preferences on n candidates with ties, but here we will only consider ones with no ties, coming from the maximal partition $\lambda = (1, 1, \dots, 1)$.

As before, for any neutral SRSF f , we can let \mathbf{s} be the vector of all $s(\cdot, r)$ and specify $f = f_{\mathbf{s}}$; then $f_{\mathbf{s}}$ is really a linear transformation $f_{\mathbf{s}} : M^{1^n} \rightarrow M^{1^n}$. Is there any *group* structure here? The answer is yes.

- By definition, SRSFs only depend on the number of each ranking in the profile \mathbf{p} (they are *anonymous*). Hence if we define $\sigma(\mathbf{p}(v)) = \mathbf{p}(\sigma^{-1}(v))$, the outcome of the SRSF will change according to σ as well. That is, the profile space M^{1^n} is a $\mathbb{Q}S_n$ -module. The same is true for the outcome space.
- But by having *neutral* SRSFs with $s(v, r) = s(\sigma(v), \sigma(r))$, we see that the action of σ propagates from profiles to outcomes. That is, for each r ,

$$\sigma \left(\sum_{v \in L(\mathcal{A})} \mathbf{p}(v) s(v, r) \right) = \sum_{v \in L(\mathcal{A})} \mathbf{p}(\sigma^{-1}(v)) s(\sigma^{-1}(v), \sigma^{-1}(r)) ,$$

which is the same as the effect of σ on the final ranking, so $f_{\mathbf{s}}$ is a $\mathbb{Q}S_n$ -module homomorphism, essentially by definition. Every single voting procedure under discussion *is* a group-theoretic object.

- We have even more; by exactly the same argument as in [7], since $\mathbf{p} \in M^{1^n} \cong \mathbb{Q}S_n$, a neutral SRSF is the result of the profile acting on \mathbf{s} , so that $f_{\mathbf{s}}(\mathbf{p}) = \mathbf{p}\mathbf{s}$, in the sense of the group rings.

Once we know that $f_{\mathbf{s}}$ is a $\mathbb{Q}S_n$ -module homomorphism, we can use representation theory to find out things about it. Our main tool will be decomposition into irreducible submodules, and the following well-known result:

SCHUR'S LEMMA. *Let G be a group. If M and N are irreducible G -modules and $g : M \rightarrow N$ is a G -module homomorphism, then either $g = 0$ or g is an isomorphism.*

Thus far, $G = S_n$ for us. It is a classical result in representation theory of the symmetric group that the irreducible modules of S_n are indexed by the partitions of n ; see the Appendix for more details and their dimensions. More importantly, the irreducible decomposition of $\mathbb{Q}S_n = M^{1^n}$ is given by the sum of a number of each of these irreducible modules, k for a k -dimensional one. For $n = 3$, this was

$$M^{(1,1,1)} \cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1,1,1)}$$

and corresponded directly with the K , B , R , and C components.

Let's use this to further justify some of the claims in Section 3. Since any $f_{\mathbf{s}}$ is a S_3 -module homomorphism, Schur's Lemma tells us that $S^{(3)}$ (K) and $S^{(1,1,1)}$ (C) go to themselves in any neutral SRSF. In both cases this will be by multiplication by some scalar. On the other hand, each $S^{(2,1)}$ can go to any linear combination of the two $S^{(2,1)}$ components – that is, B_A could go to any combination $x B_A + y R_A$.

However, these spaces are also P_n -modules, provided we restrict to procedures with reversal symmetry. Schur’s Lemma now distinguishes between the two copies of $S^{(2,1)}$ given by B and R , because $(B_X)^\rho = -B_X$ while $(R_X)^\rho = R_X$.

4.2. Voting-Theoretic Decompositions for General n . We now move to the decomposition of the profile space $M^{1^n} \cong \mathbb{Q}S_n$ as S_n - and P_n -modules. Referring to the Appendix, we note that the canonical irreducible decomposition of M^{1^n} has $n - 1$ copies of $S^{(n-1,1)}$ and $\binom{n-1}{2}$ copies of $S^{(n-2,1,1)}$. These are the only ones we will need to deal with.

There are various ways to think of $(n - 1)S^{(n-1,1)}$, but the classification in [18, 19] as the Borda (B), Alternating (Alt), and Symmetric (Sym) components is most useful. As with our $n = 3$ example, these are profile *differentials* (i.e. they are sum-zero), since they are orthogonal to $S^{(n)}$.

We summarize them in the following table, with unmentioned values being zero. Each irreducible component is the direct sum of one-dimensional vector spaces given by B_X , $Alt_{j,X}$, or $Sym_{j,X}$ (for any given j) for all candidates X .

B_X	$B_X(r) = n + 1 - 2k$	if $r(X) = k$
$Alt_{j,X}$	$Alt_{j,X}(r) = n - 1$	if $r(X) = j$
$2 \leq j \leq \frac{n}{2}$	$Alt_{j,X}(r) = 1 - n$	if $r(X) = n + 1 - j$
	$Alt_{j,X}(r) = 2j - n - 1$	if $r(X) = 1$
	$Alt_{j,X}(r) = 1 + n - 2j$	if $r(X) = n$
	$Sym_{j,X}$	$Sym_{j,X}(r) = 1$
$2 \leq j < \frac{n+1}{2}$	$Sym_{j,X}(r) = -1$	if $r(X) = 1$ or n
$Sym_{\frac{n+1}{2},X}$	$Sym_{\frac{n+1}{2},X}(r) = 2$	if $r(X) = \frac{n+1}{2}$
	$Sym_{\frac{n+1}{2},X}(r) = -1$	if $r(X) = 1$ or n

For example, when $n = 4$, the Borda component for A has 3, 1, -1 , -3 voters for rankings with A in first through fourth place respectively. That is, B_A is a profile (differential) with 3 voters each for $ABCD$, $ABDC$, $ACBD$, $ACDB$, $ADBC$, and $ADCB$, but -1 votes each for $BCAD$, $BDAC$, $CBAD$, etc., and so forth. The profile $Alt_{2,A}$ has $-1, 3, -3, 1$ for the same places¹⁴; the analogous symmetric profile grants $-1, 1, 1, -1$ votes, respectively, to rankings with A in first through fourth places.

In all cases, these have the structure that the sum over *all* candidates of each of these profiles is zero, so that each component, as a vector space, is $(n - 1)$ -dimensional. We can start to see what role these play with the following examples.

EXAMPLE 4.1. Let’s see what happens to the Borda and Sym components under plurality for $n = 4$. Recall that if $r = XYZW$, plurality is the SRSF with $\mathbf{s} = s(\cdot, r)$ giving 3 weighting points for any ranking with X in first place, 2 for Y in first, 1 for Z and 0 for W .

What happens to B_A under this system? Any ranking r of the form $AYZW$ will have 18 total compatible voters in B_A (three each of the six possible permutations with A in first place) giving 3 points each. How many voters will give 2, 1, or 0 points to r ? Once we pick Y , Z , or W to be in first place, there are two of each kind of those voters with A in second, third, and fourth place, respectively, weighted by

¹⁴So -1 votes for $ABCD$, $ABDC$, $ACBD$, $ACDB$, $ADBC$, and $ADCB$, . . .

1, -1, -3 in the profile - giving $2(1 - 1 - 3)$ ‘total voters’ giving each 2, 1, and 0 points. Subtracting this from 54 yields 36 points for r of the form $AYZW$.

In the same manner, any r of the form $XAZW$ has the same 18 voters giving *two* points each, and the rest giving 3, 1, and 0 points each, for 12 points per this type of ranking r ; by symmetry, we see that B_A will be sent to $12B_A$ by the plurality function.

EXAMPLE 4.2. Using the same strategy, the symmetric component S_A has -6 ($= -1 \cdot 6$) voters granting 3 points each to $r = AYZW$. Considering again the two rankings each with $Y, Z,$ or W first and A in the other spots, we have $2(1 + 1 - 1)$ voters giving 2, 1, and 0 points, for a total of -12 points. One can compute that rankings of the form $r = XAZW$ will receive -4 points each, and by symmetry we see that S_A is sent to $-4B_A$ by plurality.

But the $B, Alt,$ and Sym components also happen to be irreducible P_n -modules! It is not hard to see that under reversal symmetry the B and Alt components reverse sign, while the Sym components are unchanged, so we use the notation $S^{(n-1,1),+}$ for the isomorphism class of the Sym modules, while the B and Alt components are called $S^{(n-1,1),-}$.

What about generalizing the discussion for $n = 3$ to the components of $S^{(n-2,1,1)}$? Most of these vanish under even the weakest symmetry we’ll discuss, so the final component to describe corresponds to $(1, -1, 1, -1, 1, -1)$ when $n = 3$.

DEFINITION 4.3. For each pair $\{X, Y\}$ of candidates, we will define C_{XY} , the *Condorcet component*, as follows¹⁵:

Let $\{XY1\}$ denote the set of all rankings which begin $X \succ Y$, let $\{XY2\}$ denote the set of all rankings which begin with $X \succ ? \succ Y$, and continue up through $\{XY(n - 1)\}$. Then C_{XY} is the profile where, for all cyclic permutations of the elements in $\{XYi\}$ (such as ABC, BCA, CAB for three candidates), we assign $n - 2i$ voters to those rankings.

Notice that $\{XYi\}$ is simply the reversal of all $\{XY(n - i)\}$, so there is redundancy. For $n = 3$, this does give the usual Condorcet component, while for $n = 4$ it gives the Condorcet component in the form of the C_{XYs} of Saari, as for $i = 2$ we get zero voters. A convenient basis of dimension $\binom{n-1}{2}$ is given by

$$C_{A_1A_2}, C_{A_1A_3}, \dots, C_{A_1A_{n-1}}, C_{A_2A_3}, \dots, C_{A_2A_{n-1}}, \dots, C_{A_{n-2}A_{n-1}}$$

where one notes that holding X or Y constant and summing over all candidates in the other variable gives zero.

4.3. Pairwise Compatibility. In order to pursue the finest-grained results, we need one last set of concepts. The first three are directly from [7].

DEFINITION 4.4. We define the *pairs map* $P : M^{1^n} \rightarrow M^{1^n}$ to be the linear transformation that sends a basis vector of M^{1^n} to the sum of all such vectors whose top two candidates are in the same order as in the input vector.

¹⁵This is obviously indebted to Saari’s original Condorcet component and Zwicker’s ‘spin’ component [33] as well as Saari’s C_{XY} in Section 6 of [18]. Indeed, the spaces only differ when $n \geq 5$, which is probably why they are first completely described here. See also Sections 4.4.3 and 4.5 of [20] where they are implicit in a discussion about the ‘old’ Condorcet components.

For instance, if $\mathbf{p}(BACD) = 1$ and $\mathbf{p}(XYZW) = 0$ for all other rankings, $P(\mathbf{p})(XYZW) = 1$ if XY is one of the (ordered) pairs BA, BC, BD, AC, AD, CD and $P(\mathbf{p})(XYZW) = 0$ otherwise. This naturally encodes all the usual information we associate with comparing candidates on a pairwise basis – for instance, in the Borda Count and Kemeny Rule.

DEFINITION 4.5. The *effective space* of a linear transformation T is the orthogonal complement to the kernel of T . This determines what will *not* be in the kernel.

In essence, this is the subspace of elements of the domain of T (and in our context, f) which have no part simply sent to zero (in our context, a complete tie). Since we can often compute the dimension of this space, it will help us compute the dimensions of the sets of procedures. In particular (see the discussion before Theorem 6 of [7]), the effective space of P is isomorphic as an S_n -module to $S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1,1)}$. In a moment, this space will be decomposed into components we already know about.

DEFINITION 4.6. We say that any procedure whose kernel contains the kernel of P (or, what is the same, its effective space is contained in the effective space of P) is *compatible with pairs*.

Any SRSF compatible with pairs will automatically send all complete head-to-head tie portions of the profile to zero. For $n = 3$, we saw that this meant it sent R and K to zero.

But which SRSFs are compatible with pairs? Certainly BC and KR are. But there are other schemes compatible with pairs (like the Condorcet, Simpson, Dodgson, and Copeland rules) which are not SRSFs; they are not even the same as ‘composite ranking scoring functions’, which take SRSFs and break ties with other SRSFs (see [5] for details on Copeland, for instance). The next example shows a simple example which *does* fall in this category.

EXAMPLE 4.7. In Figure 9 recall the rule from Example 3.10, now with sum-zero \mathbf{s} . This vector of weights has rotational *anti*-symmetry, but R_A has 180 degree

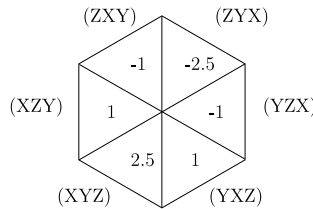


FIGURE 9. A procedure between Borda and Kemeny

rotational symmetry; hence, the procedure sends $R_A = (1, 1, -2, 1, 1, -2)$ to zero.

DEFINITION 4.8. We say that a neutral SRSF $f_{\mathbf{s}}$ *fully preserves pairs* if, as a linear transformation, it sends the subspace $S^{(n-1,1)} \oplus S^{(n-2,1,1)}$ from pairs compatibility to *exactly the same subspace*.

The subspace in the definition must be sent to an *isomorphic* subspace (or zero), by Schur’s Lemma. However, in general it can be sent to *any* isomorphic component

– for instance, the basis vector of a Borda component could be sent to some unusual linear combination of basis vectors of Alternating and Symmetric components. *But such components are themselves full pairwise ties.* So if a procedure fully preserves pairs, then the pairwise information from the Borda component is in some sense not ‘wasted’ – a key ingredient in the statement of Theorem 5.12.

EXAMPLE 4.9. There are rules which fully preserve pairs while not in fact being compatible with pairs, such as any positional scoring rule other than the Borda Count. Using plurality in Example 4.1, the *Sym* component was not killed; still, the Borda component is sent to a scalar multiple of itself (and the Condorcet component to zero).

EXAMPLE 4.10. Using the same \mathbf{s} as in Example 4.7, we can explicitly compute where B and C are sent. In fact, this SRSF happens to send C to itself, not even to a scalar multiple; KR *doubles* the influence of the Condorcet component. Similarly, the influence of B_X is multiplied by seven (by eight for BC and by six for KR).

In general, an \mathbf{s} from the general continuum in Figure 7 sends B_X to $(2a+2)B_X$ and sends C to $(2a-4)C$.

5. Theorems and the Borda-Kemeny Spectrum

5.1. Statements of Theorems. We are now ready to state our results.

The first theorem is analogous to Theorem 5 in [22], and generalizes Theorem 6 of [7].

THEOREM 5.1. *The effective space of any neutral SRSF which is compatible with pairs is $S^{(n)} \oplus B \oplus C$, where the latter two are the Borda and Condorcet spaces defined above. Its image might lie in any piece of $S^{(n)} \oplus (n-1)S^{(n-1,1)} \oplus \binom{n-1}{2}S^{(n-2,1,1)}$, however.*

This can be proved most easily by noting that the KR and BC both kill complete head-to-head ties, but take B and C to multiples of themselves (hence B and C must be the specific copies of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ in question).

PROPOSITION 5.2. *The BC sends any Borda component B_X to a multiple of B_X .*

PROPOSITION 5.3. *The KR sends any Condorcet component C_{XY} to a multiple of C_{XY} .*

Recall from Section 4.2 (see also Theorem 7.1) that the irreducible isomorphism classes $S^{(n-1,1),\pm}$ and $S^{(n-2,1,1),\pm}$ are P_n -modules, essentially differing in the same way that $S_{1,2}^{(2,1)}$ differed in Proposition 3.8. So if we look up the size of these pieces in the decomposition of Theorem 7.1 (in the Appendix) and note that $B \cong S^{(n-1,1),-}$ and $C \cong S^{(n-2,1,1),-}$ in that notation, we have:

THEOREM 5.4. *With reversal symmetry obeyed, however, the image of a neutral SRSF compatible with pairs must be in a space isomorphic to $S^{(n)} \oplus \frac{1}{2} \left(n-1 + \left(\frac{1+(-1)^n}{2} \right) \right) S^{(n-1,1),-} \oplus \frac{1}{2} \left(\binom{n-1}{2} + \lfloor \frac{n-1}{2} \rfloor \right) S^{(n-2,1,1),-}$.*

What about positional scoring rules? We state the SRSF analogue of the remark before Theorem 4 in [7] (the proof is essentially the same).

THEOREM 5.5. *The effective space of any SRSF which is a positional scoring rule (unless its vector of weights is sum-zero) is isomorphic to $S^{(n)} \oplus S^{(n-1,1)}$, where the $S^{(n-1,1)}$ component may be any piece of the whole $(n-1)S^{(n-1,1)}$ piece of $\mathbb{Q}S_n$.*

In fact, the copy of $S^{(n-1,1)}$ will depend on the structure of \mathbf{s} , as pointed out there. Nonetheless, the SRSF point of view is quite enlightening.

THEOREM 5.6. *With reversal symmetry obeyed, a positional scoring rule must have its effective space be $S^{(n)} \oplus S^{(n-1,1),-}$.*

This certainly makes sense, and this would also directly impact \mathbf{s} , as the weights $\mathbf{w}(i) + \mathbf{w}(n+1-i)$ must be invariant with respect to i in that case, or to equal zero if $S^{(n)} \rightarrow 0$.

In the next few more propositions, these techniques reprove that scoring rules using pairwise information must essentially be the Borda Count – a now-standard result.

PROPOSITION 5.7. *Any profile orthogonal to $S^{(n)} \oplus (n-1)S^{(n-1,1)}$ has the property that it is sum-zero not just as a whole, but also for the subset of rankings r such that $r(j) = X$, for any j and any X .*

PROOF. The structure of the $S^{(n-1,1)}$ components are such that each has a basis of vectors \mathbf{p} such that $\mathbf{p}(v)$ is the same for all v with $v(j) = X$ (this follows immediately from the table in Section 4.2). Any vector which has value 1 for all $v(j) = X$ and zeros elsewhere is in the subspace given by the sum of the (one-dimensional) $S^{(n)}$ subspace and the right $S^{(n-1,1)}$ component. A profile orthogonal to this subspace must necessarily fulfill both requirements of the proposition. \square

As a result, the score allocated to ranking r from any profile orthogonal to $S^{(n)} \oplus (n-1)S^{(n-1,1)}$ will be

$$\sum_{k=1}^n \left(\sum_{i=1}^{n-1} \left((n-i) \sum_{v(k)=X} t(v, r(i)) \right) \right)$$

where the innermost sum is counted with (possibly negative) multiplicity, and hence must be zero by the proposition. This leads us to the generalization of what we discovered for plurality with $n = 4$ in Example 4.1:

PROPOSITION 5.8. *The image of any positional scoring rule will be in $S^{(n)} \oplus B$.*

Just as moving to the algebraic viewpoint gives us cardinal, not just ordinal, information, this gives us more information than before. Namely, positional scoring rules are extremely limited in their outcome potential; depending on your point of view, this might be good or bad. It certainly limits the types of ties one can have, for instance; it also means a lot of complete pairwise tie information (such as the *Sym* components!) is being interpreted dubiously.

Since the intersection of $(n-1)S^{(n-1,1)}$ and $B \oplus C$ is B , we now have the following, which extends Theorem 6 of [7] (itself an extension of various results of Saari) to the SRSF context.

COROLLARY 5.9. *An SRSF which is both a scoring rule and relies only on pairwise information has an effective space and image of $S^{(n)} \oplus B$. This must be essentially the same as the Borda count (or its reversal).*

We are now ready to state and prove the main theorem.

5.2. The Borda Count and the Kemeny Rule. Even if a neutral SRSF has lots of nice properties, there are still weird things that can happen, as the following two examples demonstrate.

EXAMPLE 5.10. Assume $n = 4$, and create the reversal-symmetry-obeying SRSF which sends B_X to Alt_X . Alt_X has -1 voters for each ranking with X first, $+1$ voters for each with X last, $+3$ for each ranking with X second, and -3 for each ranking with X third. Given that B_X expresses very strong support for X , a procedure which interprets this as support for X in second place seems problematic.

EXAMPLE 5.11. Perhaps a procedure which kills off the Condorcet component *and* obeys reversal symmetry would be better. But with $n = 4$, one can construct just such an SRSF which sends B_A to $-40Alt_A$, going from overwhelming approval for A over all others to overwhelming approval for any profile with A in third place, some for ones with A in first, but the least for those with A in second! The vector \mathbf{s} would have $s(ABCD, ABCD) = 0$, $s(BACD, ABCD) = -2$, but $s(ACBD, ABCD) = 3$, and $s(BDAC, ABCD) = -5$.

Given the vector of weights \mathbf{s} , this is not a procedure one would actually use – but that is not the point. Just as in Saari’s papers [18, 19], the point is that *any* neutral, reversal-symmetric SRSF $f_{\mathbf{s}'}$ such that \mathbf{s}' had a component of this \mathbf{s} in it would incorporate some of that strange behavior.

Given the desire to avoid the behavior in the preceding examples, one must seriously consider the remaining alternatives; this is the essence of the algebraic point of view of voting theory. Once we have bothered to *get* a reasonable effective space of profiles by relying only on pairwise information, we will probably want to send that effective space to itself. This is the point of combining compatibility with pairs with the property of fully preserving pairs, and of the main theorem of the paper.

THEOREM 5.12. *Suppose a neutral SRSF is compatible with pairs and fully preserves pairs. Then (up to essential difference) this SRSF is on a one-dimensional continuum of procedures; this is precisely the continuum of procedures given by the Borda Count and Kemeny Rule.*

If in addition the Condorcet component goes to zero, the rule is the Borda Count.

We call this continuum the *Borda-Kemeny Spectrum*.

To prove Theorem 5.12 is mostly computation. By applying the additional hypotheses to Theorem 5.1 (which kills $S^{(n)}$), we see that such an SRSF must go from $B \oplus C$ to itself, hence the space is one-dimensional up to essential difference. To prove that the BC and KR are in fact on this continuum, simply collate Propositions 5.2 and 5.3, Corollary 5.9, and the following result.

PROPOSITION 5.13. *The KR sends any Borda component B_X to a multiple of B_X .*

5.3. The Borda-Kemeny Spectrum. Why might one be interested in such methods and procedures? Let’s begin with some fairly concrete computations.

For convenience, we take $s(r, r) = 1$ and $s(r, \rho(r)) = -1$. For $n = 3$ the continuum with parameter t takes the shape¹⁶ $(1, t, -t, -1, -t, 1)$, with $t = 1/3$ being Kemeny and $t = 1/2$ being Borda.

For $n = 4$ the continuum is more subtle. Assuming again that $s(r, r) = 1$ and $s(r, \rho(r)) = -1$, for a ranking r which is one neighbor swap away from v (as in the comment after Definition 2.3), we would have $s(v, r) = 2t$. For most r at distance two we would have t points, but $s(XYZW, YXWZ) = 4t - 1$. We would have $s(XYZW, YWXZ) = s(XYZW, ZXWY) = 0$, but the others at distance three having $\pm(3t - 1)$, and those further away having negative points. The Kemeny Rule is at $t = 1/3$, the Borda Count at $t = 2/5$.

Notice that this spectrum already has more complexity; for instance, $4t - 1$ could be greater than or less than t depending on whether t was greater than $1/3$ or not. As a result, it is much more difficult combinatorially to obtain sharp outcomes; here is a useful one based on a weak Pareto-type condition.

PROPOSITION 5.14. *If we assume that the partial order on the permutahedron is respected by \mathbf{s} , then we must have $1/4 \leq t \leq 1/2$.*

EXAMPLE 5.15. What is particularly interesting about this is that there are reasonable (from this point of view) methods both ‘between’ KR and BC, but also on either side of them along the continuum! For instance, if $t = 1/4$, the partial order is respected but the C_{XY} Condorcet components are sent to (small scalar multiples of) $-C_{XY}$. This is an intriguing possibility if one wanted a procedure that intentionally *controverted* the expectations of cyclic profiles slightly.

Those who have studied voting methods from the linear-algebraic or geometric perspective have usually advocated for real-life use of the Borda Count based on its intense symmetry – especially since it takes cycles like $A \succ B \succ C \succ A$ and treats them as complete ties. For instance, [7] was motivated by trying to find analogues to BC for partial ranking information. For methods intended to provide a winner or set of winners, this seems reasonable to do, even if it might violate the Condorcet criterion.

Theorem 5.12 is significant because we now have a broader range of options for the Condorcet component in symmetric procedures. The Borda Count is the dividing line between Condorcet components being sent to themselves or their negatives, so one might reject social preference functions ‘beyond’ it (for $n = 4$, with $t < 2/5$) like the one in Example 5.15. But for procedures ‘beyond’ KR, it is not clear that there is an upper bound on how much influence should be given to the Condorcet components; one might want to approximate voting on a cyclic order itself. The end of Section 4 of [35] reports that when $n = 8$ the KR and BC give radically different outcomes; so the spectrum gains even more importance.

The spectrum (and these methods) should be useful in considering manipulation. It is ‘classical’ (originally due independently to Gibbard and Satterthwaite; see [27] for a comprehensive survey) that situations exist in nearly any choice system where a voter can cast a vote other than his or her actual preference and come out with a more preferred result. Geometric-algebraic methods have been of use in analyzing BC and KR (see for instance [21]); a taste of similar analysis is demonstrated next. (The proof is exceedingly tedious, but straightforward.)

¹⁶This presentation differs slightly from before so it is easier to compare with $n = 4$.

PROPOSITION 5.16. *Given a Borda-Kemeny spectrum method with vector of weights $\mathbf{s} = (1, t, -t, -1, -t, t)$ and a profile $aB_A + bB_B + cC$ (a, b, c constants), the precise border where manipulation can happen is not just when $a = 2b$ (between ABC and BAC), but also when $b = -\frac{1-2t}{t+1}c$ (between ABC and ACB) (The no-show paradox can occur when $-\frac{1}{2} < 2a - 2b + c < 0$ with KR.)*

Let us return to the ideas of Section 1.1. Choice theory is about *choice*, not just winners. In a situation of a board of directors, it is entirely reasonable for voters for $ABCD$ to say that they would prefer $BCAD$ to $ADCB$ as an outcome; with $BCAD$ at least most of the succession is preserved, whereas with $ADCB$ their first-choice candidate wins but the rest of it is counter to their preference. We've already mentioned why it might be appropriate to leave a component of a profile which looks like $(1, 0, 1, 0, 1, 0)$ as a tie between the *rankings* ABC, BCA, CAB rather than a tie between *candidates* A, B, C .

There is one additional possible interpretation of this worth considering. The permutahedron is an abstract combinatorial object, but it may be embedded consistently in space in many ways. Zwicker has pointed out (in [35]; see also [24]) that one of the equivalent weighting vectors \mathbf{s} for both KR and BC come from square distances between its vertices in different embeddings. Might there be a way to think of some of the other methods along this spectrum as part of the continuum stretching (for $n = 3$) the regular hexagon to the permutahedral vertices of the cube? (And if so, can we find a geometric interpretation of $t > 1/2$?) Ideally, this would give a natural connection to the representation theory as well – and the combinatorial structure has given us the tools.

6. Looking Forward

6.1. Algebra in the Service of Choice. Although it is a highlight, the Borda-Kemeny Spectrum is only part of the story; the SRSFs discussed are a good starting point, but not an end in themselves. Where might the types of thinking in this paper lead us?

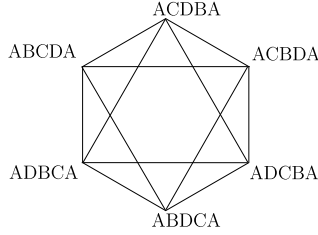
EXAMPLE 6.1. Suppose that the objects of voting really *are* cyclic orders; that is, each voter is allowed to select a cycle such as $A \succ B \succ D \succ C \succ A$. Natural contexts for this include:

- Seating preferences around a round table
- Rotating long-term site visit schedules for observation or inspection
- Scheduling space in a 24-hour facility

The combinatorial object which represents these in the same way the permutahedron represents rankings is called the *cyclic-order graph*¹⁷. With only three objects, it is nearly trivial (the only cyclic orders are $A \succ B \succ C \succ A$ and $A \succ C \succ B \succ A$, and these are equivalent if we add rotational/reversal symmetry). However, even with four ‘candidates’, there are six configurations – see Figure 10.

This graph is the skeleton of the octahedron, which has symmetry group $S_4 \times C_2$ (for reasons unrelated to the permutahedron symmetry group). The ‘profile space’ is only six-dimensional, though, not even close to the size of the order of the symmetry group. As a result, the irreducible components of the profile space are quite different.

¹⁷See [28].

FIGURE 10. The cyclic order graph CO_4

One is the trivial component where all orders receive the same number of votes. There is also a two-dimensional one, roughly equivalent to the *Sym* component for SRSFs, which is generated by a profile differential with two votes for $ABCD A$ and its reversal, and -1 votes for each of the others. The third component is *three-dimensional* and is reminiscent of the Borda component – it is generated by profile differentials with one voter for a cyclic order, and minus one voter for its reversal.

If we focus on keeping only the ‘Borda-like’ component, we discover the space of possible procedures does *not* have symmetry around any square (4-cycle) in the graph. Instead, the weighting vectors look like $(a, b, c, -b, -c, -a)$, where the $-a$ is at the reversal of the cyclic order for a .

This example exemplifies our point of view in this paper. It is not possible to justify the more obvious sum-zero vectors $(a, 0, 0, 0, 0, -a)$ without introducing additional arguments – just like we had to introduce compatibility with pairs to focus attention on the most interesting procedures. The voting theory informed the algebra.

At the same time, there is some real voting theory, not just algebra, waiting to be done! What are natural non-algebraic conditions for ‘nice’ voting systems on cyclic orders? How do people really choose to sit around a table? And does person X really care if person Y is at her right or left, as long as they are sitting close together? (In this context, it does matter – a different graph would ask what happens if it doesn’t matter.) These are all questions that require input from the choice community – just as finding new, appropriate questions to ‘ask’ the Borda-Kemeny spectrum SRSFs will take some time.

6.2. Future Work and Acknowledgments. There are many opportunities for further work here.

- What are the natural generalizations of Pareto and unanimity in the context of the permutahedron, and what properties would they imply? (This is not obvious.)
- The continuum of procedures can be different from BC and KR – how different? To what extent do they share desirable properties – especially for $n \geq 5$ (see [35])?
- What about truncated, tied, or incomplete preferences in this context?
- Cyclic order graphs, let alone representations of their automorphism groups, have not been studied much beyond [28]; what can we learn about voting in this context?
- What about voting with respect to the symmetries of some *arbitrary* graph on a set of alternatives?

- Can one give an explicit geometric model for the spectrum, as toward the end of Section 5.2?

6.2.1. *Acknowledgments.* Before acknowledging humans, I wish to explicitly point out that mathematical software (I used Sage [26]) was essential to discovering these rather subtle patterns, particularly when it came to more than $n = 4$ candidates. As Archimedes pointed out¹⁸, it is much easier to prove something once you know what to prove! Thanks also go to:

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7. Appendix

7.1. Representations of the Permutahedron. Our main goal for this section is stating and proving Theorem 7.1 about precise irreducible decompositions of $\mathbb{Q}X$. We also collate several results about the representation theory of P_n and S_n , providing proofs where these are not well-known. The book [13] is a canonical reference, but we echo the notation from the more closely related [8].

It is classical that the irreducible representations of S_n (over a field of characteristic zero) are classified by partitions λ of n , each called S^λ . For instance, for $n = 4$ there are precisely five, labeled $S^{(1,1,1,1)}$, $S^{(2,1,1)}$, $S^{(2,2)}$, $S^{(3,1)}$, and $S^{(4)}$. The regular representation $\mathbb{Q}S_n$ decomposes, as a S_n -module, as $\bigoplus_\lambda \dim(S^\lambda)S^\lambda$.

The regular representation of S_n is given by the action of S_n on the vector space $\mathbb{Q}X$, where X is the set of permutations of $\{1, 2, \dots, n\}$. But considered as the set of vertices of the permutahedron, there will be a $P_n \cong S_n \times C_2$ action on X , giving $\mathbb{Q}X$ a P_n -module structure as well.

Since P_n has such a nice structure, we know (see e.g. [1], Example 15.2) that each S^λ will be isomorphic (as an S_n -module) to two different irreducible P_n -modules, which we will call $S^{\lambda,+}$ and $S^{\lambda,-}$ to indicate how ρ acts on them (namely, $\rho S^{\lambda,+} = S^{\lambda,+}$ but $\rho S^{\lambda,-} = -S^{\lambda,-}$).

It turns out that most of this decomposition of $\mathbb{Q}X$ lies in the kernel from the perspective of voting theory. The important pieces are the $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ components, which are the ones pairwise-respecting procedures and points-based procedures are affected by.

THEOREM 7.1. *For $n > 3$, the decomposition of $\mathbb{Q}X$ as a P_n -module includes exactly the following number of copies of these irreducible submodules:*

¹⁸With respect to both Democritus and Eudoxus deserving credit for showing that a cone or pyramid has one-third the volume of the respective cylinder, see e.g. [11] for discussion.

<i>Irreducible</i>	<i>Number</i>
$S^{(n-1,1),+}$	$\frac{1}{2} \left(n - 1 - \left(\frac{1+(-1)^n}{2} \right) \right)$
$S^{(n-1,1),-}$	$\frac{1}{2} \left(n - 1 + \left(\frac{1+(-1)^n}{2} \right) \right)$
$S^{(n-2,1,1),+}$	$\frac{1}{2} \left(\binom{n-1}{2} - \lfloor \frac{n-1}{2} \rfloor \right)$
$S^{(n-2,1,1),-}$	$\frac{1}{2} \left(\binom{n-1}{2} + \lfloor \frac{n-1}{2} \rfloor \right)$

This result is all the theorems need; these numbers are given for small n in the following table.

n	$\#S^{(n-1,1),+}$	$\#S^{(n-1,1),-}$	$\#S^{(n-2,1,1),+}$	$\#S^{(n-2,1,1),-}$
3	1	1	0	1
4	1	2	1	2
5	2	2	2	4
6	2	3	4	6
7	3	3	6	9
8	3	4	9	12
9	4	4	12	16

Hence the decomposition of $M^{(n-1,1)}$ is into $\frac{n-1}{2}$ Symmetric components and $\frac{n-1}{2}$ Borda/Alternating components when n is odd, and $\frac{n}{2} - 1$ and $\frac{n}{2}$ when n is even. And the decomposition of $M^{(n-1,1,1)}$ similarly has $\binom{n-1}{2}$ total components with $\lfloor \frac{n-1}{2} \rfloor$ more of the ‘minus’ component.

Let π be the character of the P_n -module $\mathbb{Q}X$; these steps prove Theorem 7.1.

- Compute π .
- Compute the inner product (π, χ) on the space of class functions for a general character χ of an irreducible P_n -module.
- Compute (π, χ) for the specific χ we care about for the theorem.
- Apply those computations to the size of $S^{(n-1,1)}$ and $S^{(n-2,1,1)}$ to get the theorem.

For a given $g \in P_n$, the value $\pi(g)$ of the character π of $\mathbb{Q}X$ is the number of fixed points of X under that element (conjugacy class) of P_n ([1], Example 15.4). We write a generic element g as either $g = (\sigma, e)$ or $g = (\sigma, \rho)$, where $\sigma \in S_n$ and ρ is the reversing element mentioned earlier.

All vertices are fixed under the identity, so $\pi(e, e) = n!$. Since the action of a group on itself is transitive, $\pi(\sigma, e) = 0$ if $\sigma \neq e$.

For the action of the other elements of P_n , we will look more closely at what is going on. Pick an arbitrary vertex p of the permutahedron; for the purposes of the action (left or right), this should be thought of as a permutation of the set $\{1, 2, \dots, n\}$. For p to be a fixed point for $g = (\sigma, \rho)$, it must be the case that (as permutations) $p = \rho p \sigma$. That is, for each $1 \leq i \leq n$, we must have that $p(i) = n+1-p(\sigma(i))$, or $p(\sigma(i)) = n+1-p(i)$. But then $p(\sigma(i)) = n+1-p(\sigma(\sigma(i)))$ as well, so that $p(i) = p(\sigma(\sigma(i)))$, which by transitivity means $\sigma(\sigma(i)) = i$ for all i , which means σ has order two.

This narrows σ down to permutations made up of disjoint transposes (j, k) . Further, since $p(i) + p(\sigma(i)) = n + 1$, if $\sigma(i) = i$ for some i , then $p(i) = \frac{n+1}{2}$, and there can be only one such i . Hence σ is a permutation made up of as many disjoint

transposes as possible, which is the cycle decomposition type of ρ ; since the cycle decomposition type determines the conjugacy class of a permutation, σ must be in the conjugacy class of ρ ! Otherwise there are no fixed points at all.

To simplify the computation if there are, assume $\sigma = \rho$. Then any p which has $p(i) + p(n + 1 - i) = n + 1$ for all i will work. Once we have chosen $p(i)$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, that fixes the others. We can choose $p(1)$ to be anything except $\frac{n+1}{2}$ (if that is an integer), which is $2 \lfloor \frac{n}{2} \rfloor$ choices, and which then removes $p(n)$ from consideration; then $p(2)$ can be any of the remaining $2 \left(\lfloor \frac{n}{2} \rfloor - 1 \right)$ choices, and so on. Thus the number of fixed points for $g = (\rho, \rho)$ is $2^{\lfloor \frac{n}{2} \rfloor} \left(\lfloor \frac{n}{2} \rfloor! \right)$.

To summarize, $\pi(e, e) = n!$, $\pi(\rho, \rho) = \left(\lfloor \frac{n}{2} \rfloor! \right) 2^{\lfloor \frac{n}{2} \rfloor}$, and $\pi(g) = 0$ for all other elements of the group. Let $f(g)$ be the size of the conjugacy class of g . The conjugacy class of the identity is always just itself, while the conjugacy class of (ρ, ρ) is the set of all (σ, ρ) where σ have the same cycle type as ρ . The easiest way to think of such a σ is as a permutation which then has parentheses every two entries, yielding $\lfloor \frac{n}{2} \rfloor$ pairs; then we divide by the number of symmetries, of which there are 2 for each pair, and then divide by the permutations of the pairs.

Now decomposing the character π with respect to any irreducible character χ can be done directly:

$$\begin{aligned} (\pi, \chi) &= \frac{1}{2 \cdot n!} (\pi(e, e) \cdot f(e, e) \cdot \chi(e, e) + \pi(\rho, \rho) \cdot f(\rho, \rho) \cdot \chi(\rho, \rho) + 0) \\ &= \frac{1}{2 \cdot n!} \left(n! \cdot \chi(e, e) + \left(\lfloor \frac{n}{2} \rfloor! \right) 2^{\lfloor \frac{n}{2} \rfloor} \cdot \frac{n!}{\left(\lfloor \frac{n}{2} \rfloor! \right) 2^{\lfloor \frac{n}{2} \rfloor}} \chi(\rho, \rho) \right) = \frac{1}{2} (\chi(e, e) + \chi(\rho, \rho)). \end{aligned}$$

The following two propositions are enough to prove the voting assertions.

PROPOSITION 7.2. *If $\chi = \chi_{S^{(n-1,1),-}}$, then $\chi(\rho, \rho) = \left(\frac{1+(-1)^n}{2} \right)$, which is to say it alternates between 0 and 1 for n odd and even.*

PROPOSITION 7.3. *If $\chi = \chi_{S^{(n-2,1,1),-}}$, then $\chi(\rho, \rho) = \lfloor \frac{n-1}{2} \rfloor$, which is to say it goes through positive integers in order and repeats each value twice, once for n odd and once for n even.*

Before proving these statements, we finish the proof of Theorem 7.1. We already know that $\chi_{S^{(n-1,1),\pm}}(e, e) = n - 1$ and $\chi_{S^{(n-2,1,1),\pm}}(e, e) = \binom{n-1}{2}$. For the + components, the theorem is immediate. For same calculation with the - components, it suffices to recall that $\chi_{S^{\lambda,-}}(\sigma, \rho) = -\chi_{S^{\lambda,+}}(\sigma, \rho)$.

PROOF OF PROPOSITION 7.2. We look at the Borda component as being a typical example of $S^{(n-1,1),-}$. We know that $\chi(\rho, \rho)$ is the trace of the matrix given by the action of ρ on the right *and* the left of the permutahedron. We use the usual basis of $B_{A_1}, \dots, B_{A_{n-1}}$.

Conjugation by ρ is the ‘swap’ automorphism. It turns out that this sends a ranking with A_j in the i th position to one with A_{n+1-j} in the $(n+1-i)$ -th position, as we noted when calculating fixed points, where $q(i) = n+1-p(n+1-i)$. But this automatically means that B_{A_i} is sent under this action to $-B_{A_{n+1-i}}$. Combining

this with the fact that $-B_{A_n} = \sum_{i \neq n} B_{A_i}$, that means the matrix looks like

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & 0 & \dots & -1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & -1 & \dots & 0 & 0 \\ 1 & -1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

which will clearly have the correct trace. □

PROOF OF PROPOSITION 7.3. We look here at the (new) C_{XY} as a typical example of $S^{(n-2,1,1),-}$. We will use the basis mentioned above,

$$C_{A_1A_2}, C_{A_1A_3}, \dots, C_{A_1A_{n-1}}, C_{A_2A_3}, \dots, C_{A_2A_{n-1}}, \dots, C_{A_{n-2}A_{n-1}},$$

and once again will look at swapping as the action. Using the same argument as above, we see that if $A_i \succ A_j$ originally, then after swapping it is the case that $A_{n+1-j} \succ A_{n+1-i}$, so that $C_{A_iA_j} \rightarrow C_{A_{n+1-j}A_{n+1-i}}$.

The images of the basis are nearly all (different elements) in the basis too, not contributing to the trace, since our basis involves A_i, A_j where $i < j$, so $n+1-j < n+1-i$ as well. The only outlier case is when $n+1-i = n$, in which case $i = 1$, which only will contribute to the trace is if $i = n+1-j$ and $j = n+1-i$ (or $i+j = n+1$), or possibly if $i = 1$. Let's analyze this case.

When $i+j = n+1$, it contributes one to the trace. But for $0 < i < j < n$, the only pairs are for $i+j = n+1$ with $i > 1$, which means we only have to count these. So for odd n we get one pair for each integer $2 \leq i < \frac{n}{2}$, which leaves $\lfloor \frac{n}{2} \rfloor - 1$. When $i = 1$, we need to get $C_{A_{n+1-j}A_n}$ in terms of the basis, which can be rewritten as

$$- \sum_{k \neq n+1-j, n} C_{A_{n+1-j}A_k} = - \sum_{n+1-j < k < n} C_{A_{n+1-j}A_k} + \sum_{0 < k < n+1-j} C_{A_kA_{n+1-j}}.$$

This sum contributes to the trace precisely if there is a $C_{A_iA_j}$ as one of the terms, which can only happen if $n+1-j = 1$ and $k = j$, or if $k = 1$ and $n+1-j = j$. The first implies that $j = n$, which was not one of the original basis elements, but the second option implies that $j = \frac{n+1}{2}$. So if n is odd we must add one more.

Thus we arrive at a total trace of

$$\text{Tr}(\text{conj. by } \rho) = \begin{cases} \frac{n}{2} - 1 = \lfloor \frac{n-1}{2} \rfloor, & n \text{ even} \\ \lfloor \frac{n}{2} \rfloor - 1 + 1 = \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor, & n \text{ odd} \end{cases} = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

□

7.2. Proofs of Preference Function Properties.

PROOF OF PROPOSITION 5.2. Consider that B_X consists of a profile with $n+1-2k$ voters for each ranking with X in k th place. Recall that the SRSF gives $\sum_v s(v, r)$ points to ranking r , where the sum is over the whole profile (in the case of a differential, we can just subtract the negative v). We will see that the Borda Count sends B_X to a multiple of B_X , which by symmetry will mean that this is the component.

Since BC is a positional scoring rule, for a given r we have that $s(v, r) = (n - 1)t(v, r(1)) + (n - 2)t(v, r(2)) + \dots + t(v, r(n - 1))$. When we break up the sum over all v into sums over each of the subsets where $v(k) = X$, we get a double sum

$$\sum_{k=1}^n \left(\sum_{v(k)=X} (n + 1 - 2k) [(n - 1)t(v, r(1)) + (n - 2)t(v, r(2)) + \dots + t(v, r(n - 1))] \right)$$

which exhibits a very high degree of symmetry.

Assume that $X = r(j)$. Consider the two terms in the double sum above for some k and the corresponding $n + 1 - k$ (which will have opposite sign). Since B_X has the same number of voters for *all* rankings with $v(k) = X$, then for a given $v(i) = r(\ell)$, the two sets

$$\{v|v(k) = X, v(i) = r(\ell), i \neq k, n + 1 - k\} \text{ and } \{v|v(n + 1 - k) = X, v(i) = r(\ell), i \neq k, n + 1 - k\}$$

have the same size. That means that the terms in $s(v, r)$ corresponding to these sets will cancel, since they correspond to $(n - \ell)t(v, r(\ell))$ when $r(\ell) = v(i)$, and there are equal numbers of these for k and $n + 1 - k$ as long as $i \neq k, n + 1 - k$.

So, for each v such that $v(k) = r(j) = X$ and $v(n + 1 - k) = r(i)$ (where obviously $i \neq j$), the non-canceling part of the term is

$$(n + 1 - 2k) [(n - j)t(v, r(j)) + (n - i)t(v, r(i))] ,$$

which is okay since when $k = 1$ we correctly get 0 as the inside coefficient in $s(v, r)$. Sum this back up and substitute in the Borda Count values of $t(v, r(j)) = t(v, X) = \frac{n-k}{n-1}$ and $t(v, r(i)) = \frac{n-(n+1-k)}{n-1} = \frac{k-1}{n-1}$ to get

$$\sum_{k=1}^n \left(\sum_{v(k)=X=r(j), v(n+1-k)=r(i)} (n + 1 - 2k) \left[(n - j) \frac{n - k}{n - 1} + (n - i) \frac{k - 1}{n - 1} \right] \right) .$$

There are $(n - 1)!$ different v such that $v(k) = X$, and hence $(n - 2)!$ different v such that $v(k) = X$ and $v(n + 1 - k) = r(i)$ in the above sum. Then we get

$$\sum_{k=1}^n (n + 1 - 2k) \left((n - 1)! (n - j) \frac{n - k}{n - 1} + (n - 2)! \sum_{i \neq j} (n - i) \frac{k - 1}{n - 1} \right) .$$

In fact, a little clearing of denominators yields

$$(n - 2)! \sum_{k=1}^n (n + 1 - 2k) \left((n - j)(n - k) + \sum_{i \neq j} (n - i) \frac{k - 1}{n - 1} \right) .$$

The reader will notice that the sum only depends on j , as one would hope. If we increase j by one, the difference between two of these scores is

$$\begin{aligned} & (n - 2)! \sum_{k=1}^n (n + 1 - 2k) \left((n - j)(n - k) + \sum_{i \neq j} (n - i) \frac{k - 1}{n - 1} \right) \\ & - (n - 2)! \sum_{k=1}^n (n + 1 - 2k) \left((n - (j + 1))(n - k) + \sum_{i \neq j+1} (n - i) \frac{k - 1}{n - 1} \right) \end{aligned}$$

which can be simplified to a formula *not* depending on j , as needed:

$$(n-2)! \sum_{k=1}^n (n+1-2k) \left((n-k) - \frac{k-1}{n-1} \right) = (n-2)! \frac{n^2(n+1)}{6}.$$

Similarly, we need that $j=1$ and $j=n$ are opposites, and indeed

$$(n-2)! \sum_{k=1}^n (n+1-2k) \left((n-1)(n-k) + \sum_{i \neq 1} (n-i) \frac{k-1}{n-1} \right) \\ + (n-2)! \sum_{k=1}^n (n+1-2k) \left((n-n)(n-k) + \sum_{i \neq n} (n-i) \frac{k-1}{n-1} \right)$$

simplifies to

$$(n-2)! \sum_{k=1}^n (n+1-2k) \left((n-1)(n-k) + \frac{k-1}{n-1} \left((n-1) + (n-n) + 2 \sum_{i=2}^{n-1} n-i \right) \right) \\ = (n-2)! \sum_{k=1}^n (n+1-2k) \left((n-1)(n-k) + \frac{k-1}{n-1} \left(\frac{n(n-1)}{2} + \frac{(n-1)(n-2)}{2} \right) \right) \\ = (n-2)!(n-1)^2 \sum_{k=1}^n (n+1-2k) = 0.$$

□

PROOF OF PROPOSITION 5.3. For the Kemeny Rule, recall that the SRSF function is given by $s(v, r) = \sum_{a, b \in A} \delta(v, r, a, b)$, where δ is 1 if $a \succ_b$ by both rankings v and r , and is 0 otherwise; clearly this relies only on pairwise information. Let us see where it sends the (new) Condorcet components.

Using the notation above for $\{XYi\}$ we see that for a given ranking r (with $r \in \{XYj\}$) the score for r is

$$\sum_{i=1}^{n-1} (n-2i) \sum_{v \in \{XYi\}} \sum_{a, b \in A} \delta(v, r, a, b).$$

This depends only on j since the sums are over all $v \in \{XYi\}$. For each $v \in \{XYi\}$ such that $a \succ_v b$ and $a \succ_r b$, reversing r will cause δ to go from 1 to 0, but will send δ from 0 to 1 for the reversal of v ; since this reversal is in $\{XY(n-i)\}$, the score for the reversal of r is

$$\sum_{i=1}^{n-1} (n-2i) \sum_{v \in \{XY(n-i)\}} \sum_{a, b \in A} \delta(v, r, a, b) = \sum_{k=1}^{n-1} (2k-n) \sum_{v \in \{XYk\}} \sum_{a, b \in A} \delta(v, r, a, b)$$

which is a change of sign, as expected.

Furthermore, if r changes from $\{XYj\}$ to $\{XY(j+1)\}$ via a one-position swap, then all δ values will be the same except ones concerning that pair. So each v with that pair as in r loses 1, while each with the pair as in r' gains one.

If such a swap changes things, it must involve X or Y ; we will suppose it is Y_- changing to $_Y$ (other possibilities are very similar). The number of potential places for the candidate $_$ to occur *after* Y in a ranking in the set $\{XYi\}$ varies from zero to $n-2$, but there will be two potential ones of type $n-1-i$ (such as, for

$i = 2, X_Y_ \dots _$ and $_Y_ \dots _X$). Hence there are, for $\{XYi\}$, $n - 1 - i + \sum_{i=0}^{n-2} i = n - 1 - i + \frac{(n-1)(n-2)}{2}$ possibilities out of a total of $n(n-2)$ total; the difference is

$$\begin{aligned} & n - 1 - i + \frac{(n-1)(n-2)}{2} - \left(n(n-2) - \left(n - 1 - i + \frac{(n-1)(n-2)}{2} \right) \right) \\ &= 2 \left(n - 1 - i + \frac{(n-1)(n-2)}{2} \right) - n^2 + 2n = 2n - 2 - 2i + (n-1)(n-2) - n^2 + 2n = n - 2i. \end{aligned}$$

The other $n - 3$ spots have $(n - 3)!$ different possibilities, so there are $(n - 2i)(n - 3)!$ net rankings v in $\{XYi\}$ which will lose $\delta = 1$ (which, for i for which this is negative, correspond to gaining $\delta = 1$). Thus the difference in the scores

$$\sum_{i=1}^{n-1} (n - 2i) \sum_{v \in \{XYi\}} \sum_{a,b \in A} \delta(v, r, a, b)$$

will be

$$\sum_{i=1}^{n-1} (n - 2i) [(n - 2i)(n - 3)!] = (n - 3)! \sum_{i=1}^{n-1} (n - 2i)^2 = \frac{n!}{3}.$$

□

PROOF OF PROPOSITION 5.13. We use the same decomposition as above for the Borda Count, over $v(k) = X = r(j)$. As above,

$$\sum_{k=1}^n \left(\sum_{v(k)=X} (n + 1 - 2k) \sum_{a,b \in A} \delta(v, r, a, b) \right).$$

This also only depends on j (here, r is a fixed ranking with $r(j) = X$, but nothing else known) because of the sum over all v with each $v(k) = X$. However, it is useful to focus on a specific r for proving this sends Borda to Borda.

First let us observe what happens to the score when r is reversed. For each v such that $a \succ_v b$ and $a \succ_r b$, we will get $+(n + 1 - 2k)$, depending on $v(k) = X$. But if r is reversed, these go away, and the reversal of v will have $\delta = 1$. This will exactly give the negative of the original score, because if $v(k) = X$, then the reversal v' has $v'(n + 1 - k) = X$, which means it will contribute $+(n + 1 - 2(n + 1 - k)) = -n - 1 + k = -(n + 1 - k)$ to the score.

We also need to have a fixed change in score when r changes. Let r' be the same ranking as r but with $r'(j + 1) = X$. As with the C_{XY} components, we will suppose the change is $X_$ changing to $_X$. The number of potential places for the candidate $_$ to occur *after* X in a ranking in the set $\{v(i) = X\}$ is $n - i$, and the number of places before is $i - 1$, so the difference is $n + 1 - 2i$. For the remaining spots there are $(n - 2)!$ possibilities, so there are $(n + 1 - 2i)(n - 2)!$ net rankings v in $\{v(i) = X\}$ which will lose $\delta = 1$ (or, if $(n + 1 - 2i)(n - 2)! < 0$, gain $\delta = 1$).

Thus the difference in the scores given by

$$\sum_{k=1}^n \left(\sum_{v(k)=X} (n + 1 - 2k) \sum_{a,b \in A} \delta(v, r, a, b) \right)$$

will be

$$\sum_{k=1}^n ((n + 1 - 2k)^2 (n - 2)!) = (n - 2)! \sum_{k=1}^n (n + 1 - 2k)^2 = \frac{(n + 1)!}{3}.$$

□

PROOF OF PROPOSITION 5.8. Let us see what happens to the profile \mathbf{p} which has value $\mathbf{p}(v) = 1$ for $v(j) = X$ and zeros otherwise. In this case,

$$\sum_{k=1}^n \left(\sum_{v(k)=X} \left(\sum_{i=1}^{n-1} (n-i)t(v, r(i)) \right) \right) = \sum_{v(j)=X} \left(\sum_{i=1}^{n-1} (n-i)t(v, r(i)) \right)$$

Now suppose that $r(\ell) = X$; then it makes sense to rewrite this as

$$\sum_{v(j)=X} \left((n-\ell)t(v, r(\ell)) + \sum_{i=1, i \neq \ell}^{n-1} (n-i)t(v, r(i)) \right).$$

Call the weighting vector w . Then note that in the second sum inside the parentheses, for a given i , $v(k) = r(i)$ the same number of times (to be precise, $(n-2)!$ times), other than $v(j)$, of course. So we can rewrite this

$$\begin{aligned} & \sum_{v(j)=X} \left((n-\ell)w(j) + \sum_{v(j)=X} \sum_{i=1, i \neq \ell}^{n-1} (n-i)t(v, r(i)) \right) \\ &= (n-1)!(n-\ell)w(j) + \sum_{i=1, i \neq \ell}^{n-1} (n-i)(n-2)! \sum_{k=1, k \neq j}^n w(k) \\ &= (n-2)! \left[(n-1)(n-\ell)w(j) + \sum_{k=1, k \neq j}^n \sum_{i=1, i \neq \ell}^{n-1} w(k)(n-i) \right] \end{aligned}$$

It is easy to see that the difference in this caused by changing $r(\ell) = X$ to $r(\ell+1) = X$ is $(n-2)! \left[(n-1)w(j) - \sum_{k=1, k \neq j}^n w(k) \right]$, which does not depend on ℓ . Likewise, adding the cases for $\ell = 1$ and $\ell = n$ gives

$$\begin{aligned} (n-2)! \left((n-1)^2 w(j) + \sum_{k=1, k \neq j}^n w(k) \left(\sum_{i=1}^{n-1} n-i + \sum_{i=2}^{n-1} n-i \right) \right) \\ = (n-1)!(n-1) \sum_{k=1}^n w(k) \end{aligned}$$

so that it goes to the Borda component alone if the weighting vector is sum-zero, otherwise it is ‘shifted’ by a multiple of the sum of the weights. □

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GORDON COLLEGE, 255 GRAPEVINE ROAD, WENHAM, MASSACHUSETTS 01984
E-mail address: karl.crisman@gordon.edu

Double-interval societies

Maria Margaret Klawe, Kathryn L. Nyman, Jacob N. Scott,
and Francis Edward Su

ABSTRACT. Consider a society of voters, each of whom specify an *approval set* over a linear political spectrum. We examine *double-interval societies*, in which each person's approval set is represented by two disjoint closed intervals, and study this situation where the approval sets are *pairwise-intersecting*: every pair of voters has a point in the intersection of their approval sets. The *approval ratio* for a society is, loosely speaking, the popularity of the most popular position on the spectrum. We study the question: what is the minimal approval ratio for such a society? We provide a lower bound for the approval ratio, and examine a family of societies that have rather low approval ratios. These societies arise from *double- n strings*: arrangements of n symbols in which each symbol appears exactly twice. Tweaking these examples suggests a conjecture: that the minimal approval ratio of a pairwise-intersecting double-interval society is $1/3$.

1. Introduction

Consider the voting model of Berg et. al.[1] in which a political spectrum X is viewed as a continuum, with liberal positions on the left and conservative positions on the right, and in which each voter v “approves” an interval of positions along this line. For example, a tolerant moderate might approve a wide interval near the middle of the line, while an intolerant partisan may approve a narrower interval near one of the ends.

More formally, a *society* is a spectrum X together with a set of voters V and a collection of approval sets $\{A_v\}$, one for each voter. A point on the spectrum X is called a *platform*. In our situation, we imagine X to be \mathbb{R} , and each approval set A_v is a closed interval that represents the set of all platforms that v approves.

Now suppose that every pair of people can agree on some platform; that is, their intervals overlap. In this situation, Helly's Theorem [3] implies that there exists a point on the line that lies in everyone's approval set, i.e., there is a platform that everyone approves. Thus a strong hypothesis (pairwise intersecting sets) produces a strong conclusion (a point in all the sets). However, in voting theory, we are usually not looking for unanimity, but may be satisfied with a platform that has high *approval ratio*: the fraction of voters that approve this platform.

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Various authors have relaxed the hypotheses. Berg et.al.[1] define a linear (k, m) -agreeable society in which voter preferences again are modeled by closed intervals in \mathbb{R}^1 . In this society, given any set of m voters, there exists a subset of k voters whose approval intervals mutually intersect. They prove that there must exist some platform with approval ratio $\frac{k-1}{m-1}$. Another generalization by Hardin [2] looks at approval intervals on a circle rather than a line, and finds that with (k, m) -agreeability, the approval ratio of the society is at least $\frac{k-1}{m}$.



FIGURE 1. A pairwise-intersecting society of size 4 with approval number 3.

We generalize the one-interval model to a society in which every member is identified with two disjoint approval intervals and call such a society a *double-interval society*. This situation may arise naturally in the context of voting to account for voters who do not place candidates along a linear spectrum in exactly the same order, or to account for voters who find disjoint sets of platforms appealing for entirely different reasons (e.g., for being a party purist, or having the ability to work across party lines). In a scheduling context, such intervals might model a society of workers, each of whom has two different work shifts.

Figure 1 illustrates a double-interval society with four voters. The approval sets of each voter have been separated vertically so they are easier to see. Note that the approval sets are pairwise-intersecting: each voter overlaps every other voter in one or both of their approval intervals. In this example, there are several platforms approved by three voters, but no platform is approved by all four. The *approval number* of a platform $a(p)$ is the number of voters (in a society S) who approve of platform p . The *approval number* of a society $a(S)$ is the maximum approval number over all platforms in the spectrum X . That is

$$a(S) = \max_{p \in X} a(p).$$

Finally, define the *approval ratio* of a society to be the approval number of S divided by the number of voters in S .

The main question we address in this paper is: what is the minimal approval ratio of a pairwise-intersecting, double-interval society with n voters?

Examples suggest that the minimal approval ratio of such societies is $1/3$; that is, there is always a platform that will get at least a third of the votes. Our results in this paper attempt to clarify this intuition.

We will first examine a family of double-interval societies with low approval ratios that have regular patterns of interval overlap. These arise from the construction of what we call *double- n strings*, defined in the next Section. The combinatorics of such strings are quite nifty and provide a lower bound for the approval ratio of societies in this family (Theorem 3.6) as well as an upper bound (Theorem 3.7) for societies in this family. Roughly speaking, the double- n strings produce societies with asymptotic approval ratios between 0.348 and 0.385.

We will also prove a general lower bound for the approval ratio of any pairwise-intersecting, double-interval society in Theorem 4.1, which shows the approval ratio is always greater than 0.268.

Then we ask if we can find specific societies with lower approval ratios than the ones arising from double- n strings, and discover that there are such examples. We find them by modifying the construction that comes from double- n strings. See Table 1. However, all of these examples have approval ratio equal to $1/3$.

2. Double- n String Societies

Double-interval societies with regular patterns of interval overlap can be represented by *double- n strings*, that is, strings of length $2n$ containing exactly two occurrences of each of n symbols. At times we will also represent double- n strings as strings of the symbols $1, \dots, n$. We define the *distance* between two distinct symbols in a double- n string to be the minimum distance between a pair of occurrences of the symbols, where the distance between two adjacent symbols is taken as 1. The *diameter* of a double- n string is the maximum over all $1 \leq i < j \leq n$ of the distance between i and j . We will call two entries in the list *adjacent* if their positions in the list differ by no more than the diameter of the string.

Let $\delta(n)$ be the minimum diameter over all double- n strings.

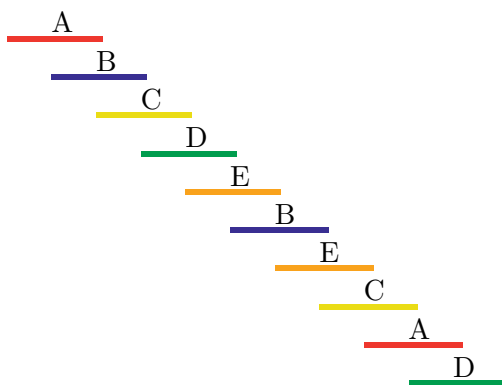


FIGURE 2. A society represented by the double-5 string $ABCDEBECAD$.

We can construct a pairwise-intersecting double-interval society from a double- n string with diameter d by assigning intervals of equal width to the symbols, long enough so that each interval overlaps the intervals of the d symbols to its right and left.

For example, consider the double-5 string $ABCDEBECAD$. This string has diameter 2, since any pair of symbols A through E appear somewhere in this list separated by at most one other symbol (e.g., the second occurrences of A and E in this string are distance 2 apart). We build a society from this string by assigning intervals of equal width as in Figure 2. This society has approval number 3 as can be seen since the right endpoint of A 's first interval intersects the left endpoint of C 's first interval, and both intersect B 's first interval. Hence we see that $\delta(5) \leq 2$ (and in fact $\delta(5) = 2$). Note that in general the approval number of a society with an underlying double- n string is one more than the diameter, that is, $a(S) = d + 1$.

3. Asymptotic approval ratios for double- n string societies

If S arises from a double- n string with diameter d , then since $a(S) = d + 1$, we see that the minimal approval ratio of such a society is $(\delta(n) + 1)/n$. By taking limits, we see that

$$\Delta = \lim_{n \rightarrow \infty} \frac{\delta(n) + 1}{n} = \lim_{n \rightarrow \infty} \frac{\delta(n)}{n}$$

is the asymptotic approval ratio for societies arising from double- n strings. In this section, we will show that

$$8/23 \leq \Delta \leq 5/13.$$

It is clear that for $n > 1$ we have $\delta(n - 1) \leq \delta(n)$ since for any double- n string we can form a double- $(n - 1)$ string of no larger diameter by deleting both occurrences of the n -th symbol. Given a double- n string S we label the symbols as $1, 2, \dots, n$ according to the left to right order of their first occurrence within S .

It is easy to see that $\Delta \leq 1/2$ since the double- n string $1, 2, \dots, n, 1, 2, \dots, n$ shows that $\delta(n) \leq n/2$ for n at least 2. In fact, we can show $\delta(n) < n/2$ for n at least 3. Although this does not change the upper bound on Δ , we will need this result for the lower bound $8/23 \leq \Delta$.

LEMMA 3.1. *If $n > 2$, then $\delta(n) < n/2$.*

PROOF. The case $\delta(3) = 1$ follows from the double-3 string $1, 2, 3, 1, 2, 3$ and the case $\delta(4) = 1$ follows from the double-4 string $1, 2, 3, 4, 1, 3, 2, 4$. The proof for the general case $n > 4$ is based on this double-4 string. Let $r = \lfloor n/4 \rfloor$. We will partition $1, 2, \dots, n$ into four strings S_1, S_2, S_3, S_4 where each is of length r or $r + 1$ depending on the value of $n \bmod 4$. We prove the result by looking at the diameter of the double- n string $T(n) = S_1, S_2, S_3, S_4, S_1, S_3, S_2, S_4$.

Suppose $n = 4r$ for some positive integer r . We let S_i be the string $(i - 1)r + 1, (i - 1)r + 2, \dots, ir$ of length r and it is easy to check that the diameter of $T(n)$ is $2r - 1$ and we have $2r - 1 < n/2$. For $n = 4r + 1$, let S_1, S_2, S_3 be as before and let S_4 be the string $3r + 1, \dots, 4r + 1$ of length $r + 1$. Now the diameter of $T(n)$ is $2r$, and again we have $2r < n/2$ as desired. For $n = 4r + 2$ we set S_1 and S_4 to have length r and S_2 and S_3 to have length $r + 1$. Consider b in S_i and c in S_j with $i < j$. It is easy to see that unless $i = 2$ and $j = 3$, the distance between b and c in $T(n)$ is at most $2r$ since at least one of S_i and S_j has length r and the other has length at most $r + 1$. Moreover, for b in S_2 and c in S_3 the distance between b and c in $T(n)$ is at most $r + 1$ since both of the substrings S_2, S_3 and S_3, S_2 occur in $T(n)$. Thus the diameter of $T(n) = 2r$ and we have $2r < 2r + 1 = n/2$. Finally for $n = 4r + 3$ we set S_1 to have length r and S_2, S_3, S_4 to have length $r + 1$. In this case it is easy to see that the diameter of $T(n)$ is $2r + 1$ and we have $2r + 1 < 2r + 3/2 = n/2$. \square

For simplicity, without loss of generality assume that if the first occurrence of symbol m occurs at position i in a double- n string, then all symbols at positions $1 \leq j < i$ are less than m (otherwise this condition can be satisfied by a permutation of the symbols in the double- n string). From Lemma 3.1, it is sufficient to consider double- n strings that have diameter less than $n/2$. It is also easy to obtain the lower bound $\Delta \geq 1/3$ as shown in the following lemma.

LEMMA 3.2. *Let r be a positive integer. We have $\delta(3r + 1) \geq r$.*

PROOF. Let $n = 3r + 1$. In any double- n string of diameter d , the first occurrence of the symbol 1 can be adjacent to at most d other symbols while the second occurrence can be adjacent to at most $2d$. Because 1 must be adjacent to all $n - 1$ other symbols, $d + 2d \geq n - 1 = 3r$, and so $d \geq r$. \square

LEMMA 3.3. *In a double- n string with diameter d , the first $n - d$ symbols are distinct (and hence in the order $1, 2, \dots, n - d$).*

PROOF. Assume that there exists some symbol x both of whose occurrences are within the first $n - d$ entries. Thus the first occurrence of n must be at position at least $n + 1$, so the distance between x and n is at least $d + 1$, a contradiction. \square

LEMMA 3.4. *Let $d < n/2$ be the diameter of a double- n string, and let r_i be the number of symbols both of whose occurrences are within d of either occurrence of i for $1 \leq i \leq d + 1$. Then $r_i \leq 3d + i - n$.*

PROOF. As $d < n/2$, Lemma 3.3 gives that the first $d + 1$ symbols of such a double- n string are $1, 2, \dots, d + 1$. For $1 \leq i \leq d + 1$, there are only $i - 1$ symbols before the first occurrence of i , so there are at most $3d + i - 1$ symbols adjacent to i , of which r_i of them are repeats. Hence $n - 1 \leq 3d + i - 1 - r_i$. \square

COROLLARY 3.5. *For $1 \leq i \leq d + 1$, and $d < n/2$, at most $3d + i - n$ of the symbols $1, 2, \dots, \hat{i}, \dots, d + 1$, are within d of the second occurrence of i . (Here \hat{i} means omit i).*

PROOF. This follows directly from Lemma 3.4 since each of the symbols $1, \dots, \hat{i}, \dots, d + 1$ occurs within d of the first occurrence of i . \square

We are now ready to prove the lower bound.

THEOREM 3.6. *Let r be a positive integer. Then $\delta(23r) \geq 8r$. Thus the asymptotic approval ratio for double- n strings is bounded below by $8/23$.*

PROOF. Let $n = 23r$ and let S be a double- n string with diameter d . Suppose $d < 8r$. Since d is an integer we have $d \leq 8r - 1$. Note that $d \geq \delta(23r) \geq \delta(21r + 1) \geq 7r$ by Lemma 3.2.

By Lemma 3.3 the first $n - d \geq 23r - 8r + 1 = 15r + 1$ symbols in S are distinct (and in order). Now since $d < 8r$ the first occurrence of the symbol labeled $15r + 1$ is not within d of the first occurrence of i for $1 \leq i \leq 7r + 1$. Thus for any such i we must have the second occurrence of i occurring in one of three sets of positions, namely the block B_1 of length d following the first occurrence of $15r + 1$, the block B_2 of length d ahead of the second occurrence of $15r + 1$, or the block B_3 of length d following the second occurrence of $15r + 1$. These blocks are illustrated in Figure 3. Let k_j be the number of symbols in $1 \leq i \leq 7r + 1$ with their second occurrence in block B_j . From the preceding observation we have $k_1 + k_2 + k_3 \geq 7r + 1$ (conceivably such a second occurrence of i could be in both B_1 and B_2 if they overlap).

Note that any pair of symbols in B_j lie within d of each other. Suppose without loss of generality that the second occurrence of 1 lies in B_1 . For any i with $1 < i \leq 7r + 1$ with the second occurrence of i in B_1 , both occurrences of i lie within d of an occurrence of 1, since $d \geq 7r$. By Corollary 3.5, the number of such i is at most

$$3d + 1 - n \leq 3(8r - 1) + 1 - 23r = r - 2,$$

giving $k_1 \leq 1 + r - 2 = r - 1$.

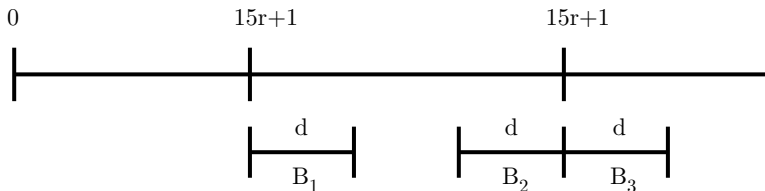


FIGURE 3. B_1 , B_2 , and B_3 (in the case that they are disjoint).

Let x be the minimal number such that the second occurrence of x is not in B_1 . Then $x \leq r$ since $k_1 \leq r - 1$. Without loss of generality suppose the second occurrence of x is in B_2 . Again, by Corollary 3.5 there are at most

$$3d + r - n \leq 3(8r - 1) + r - 23r = 2r - 3$$

symbols i with $1 \leq i \leq 7r + 1$ other than x in B_2 , so $k_2 \leq 2r - 2$.

Similarly, let y be the smallest symbol (in value) whose second occurrence is in B_3 (i.e., is not in B_1 or B_2). There are at most $k_1 + k_2$ symbols in $B_1 \cup B_2$, so $y \leq 3r - 2$. Using Corollary 3.5 one last time, we see that there are at most $3d + (3r - 2) - n \leq 3(8r - 1) + (3r - 2) - 23r = 4r - 5$ symbols $i \neq y$ with $1 \leq i \leq 7r + 1$ in B_3 , so $k_3 \leq 4r - 4$. However, this is a contradiction: we needed $k_1 + k_2 + k_3 \geq 7r + 1$, but

$$k_1 + k_2 + k_3 \leq (r - 1) + (2r - 2) + (4r - 4) = 7r - 7.$$

Therefore we could not have $d < 8r$, proving the theorem. \square

A general argument showing $\delta(br) \geq ar$, for large r , leads to the inequalities $b < 3a$ and $23a \leq 8b$. Thus the lower bound of Theorem 3.6 is the best possible asymptotic bound using this argument. Now we turn to the upper bound.

THEOREM 3.7. *For any $n > 0$, there exists a double- n string with diameter $d \leq 5 \lceil \frac{n}{13} \rceil - 1$. Hence the asymptotic approval ratio for double- n strings is bounded above by $5/13$.*

PROOF. Note that the double-13 string

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 1, 11, 6, 12, 13, 5, 4, 7, 11, 10, 9, 2, 3, 13, 12, 8$$

has diameter 4, meaning that any two symbols in it appear somewhere in the double-13 string separated by no more than three other elements. This yields a general construction for double- n strings of any length. Let $k = \lceil \frac{n}{13} \rceil$. Then replacing each symbol i in the above string with the substring

$$k(i - 1) + 1, k(i - 1) + 2, \dots, ki,$$

and removing any symbols in the resulting string that are greater than n , yields a double- n string. An example of this string for $n = 34$ ($k = 3$) is shown in Figure 4. Because the diameter of the above double-13 string is 4, any two symbols $1 \leq i < j \leq n$ are within substrings that are separated by at most three substrings of length k . Also, i and j are at worst on the far ends of their substrings, giving a maximum total distance between i and j in the new string of

$$3 \lceil \frac{n}{13} \rceil + \left(2 \lceil \frac{n}{13} \rceil - 1 \right) = 5 \lceil \frac{n}{13} \rceil - 1. \quad \square$$

(1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18)(19, 20, 21)(22, 23, 24)
 (25, 26, 27)(28, 29, 30)(1, 2, 3)(31, 32, 33)(16, 17, 18)(34)(13, 14, 15)(10, 11, 12)
 (19, 20, 21)(31, 32, 33)(28, 29, 30)(25, 26, 27)(4, 5, 6)(7, 8, 9)(34)(22, 23, 24)

FIGURE 4. A double-34 string with diameter ≤ 14 constructed as in Theorem 3.7. Symbols are grouped together by parentheses to elucidate its construction. Some groupings have fewer than three elements since symbols larger than 34 in value are removed. Empty groupings are also omitted.

4. A double-interval society lower bound

In the previous section we considered double- n strings as examples of societies with low approval ratios. These examples give upper bounds for the minimal guaranteed approval ratio for any society. In this section, we give lower bounds for the minimal guaranteed approval ratio, by considering how general pairwise-intersecting double-interval societies force conditions on the number of intervals that can intersect a given interval at its endpoints. This approach largely ignores the geometry of the approval sets and considers only combinatorial constraints.

THEOREM 4.1. *The approval number $a(S)$ of any n -voter society S satisfies*

$$(4.1) \quad a(S) \geq \left\lfloor 2n + \frac{1}{2} - \sqrt{3n^2 - n + \frac{1}{4}} \right\rfloor.$$

Then the approval ratio satisfies

$$(4.2) \quad \frac{a(S)}{n} \geq 2 - \sqrt{3} + \frac{3 + \sqrt{3}}{6n} - \frac{\sqrt{3}}{24n^2} \approx 0.268 + \frac{0.789}{n} - \frac{1.732}{24n^2}.$$

Alternatively, the size n of a society achieving a given approval number $a(S)$ is bounded above by

$$(4.3) \quad n \leq \left\lfloor 2a(S) - \frac{3}{2} + \sqrt{3(a(S))^2 - 5a(S) + \frac{9}{4}} \right\rfloor.$$

PROOF. Let A_i and A'_i represent the left and right intervals, respectively, of voter i 's approval set in the n -voter society S . Without loss of generality we may assume no two interval endpoints coincide. For any interval I , define numbers $L(I)$, $R(I)$, $B(I)$, and $C(I)$ to keep track of the number of other intervals that intersect I in various ways. Let $L(I)$ count the number of other intervals that, of two endpoints of I , contain only the *left* endpoint. Let $R(I)$ count the number of other intervals that, of two endpoints of I , contain only the *right* endpoint. Let $B(I)$ count the number of other intervals that contain *both* endpoints of I . Let $C(I)$ count the number of other intervals that intersect I but contain *neither* endpoint of I , and are hence in the “center” of I .

For example, in Figure 1, we see that $L(A') = 2$, $R(A') = 0$, $C(A') = 3$, and $B(A') = 0$. Also $L(C') = 0$, $R(C') = 1$, $C(C') = 0$, and $B(C') = 1$. Since each set must intersect all $n - 1$ other sets,

$$L(A_i) + L(A'_i) + R(A_i) + R(A'_i) + C(A_i) + C(A'_i) + B(A_i) + B(A'_i) \geq n - 1.$$

Then clearly

$$(4.4) \quad \sum_{i=1}^n [L(A_i) + L(A'_i) + R(A_i) + R(A'_i) + C(A'_i) + C(A_i) + B(A_i) + B(A'_i)] \geq n(n-1).$$

Note that an interval J covers both endpoints of another interval I and contributes 1 to the count $B(I)$ exactly when I is in the center of J and contributes 1 to the count $C(J)$. This implies:

$$(4.5) \quad \sum_{i=1}^n [B(A_i) + B(A'_i)] = \sum_{i=1}^n [C(A_i) + C(A'_i)].$$

Notice that given an approval number $a(S)$, each interval may have at most $a(S) - 1$ other sets intersecting its left endpoint. This gives an initial bound

$$\sum_{i=1}^n [L(A_i) + L(A'_i) + B(A_i) + B(A'_i)] \leq 2n(a(S) - 1).$$

and similarly, considering right endpoints:

$$\sum_{i=1}^n [R(A_i) + R(A'_i) + B(A_i) + B(A'_i)] \leq 2n(a(S) - 1).$$

However, if the $2n$ intervals are ordered by the left endpoint, then the k th interval under this ordering from left to right can have at most $k - 1$ intervals intersecting its left endpoint, not $a(S) - 1$. Thus we need to adjust the formulas above, to obtain:

$$\sum_{i=1}^n [L(A_i) + L(A'_i) + B(A_i) + B(A'_i)] \leq 2n(a(S) - 1) - \frac{a(S)(a(S) - 1)}{2},$$

$$\sum_{i=1}^n [R(A_i) + R(A'_i) + B(A_i) + B(A'_i)] \leq 2n(a(S) - 1) - \frac{a(S)(a(S) - 1)}{2}.$$

Adding these equations and applying equation (4.5) yields

$$\sum_{i=1}^n [L(A_i) + L(A'_i) + R(A_i) + R(A'_i) + C(A'_i) + C(A_i) + B(A_i) + B(A'_i)] \leq 4n(a(S) - 1) - a(S)(a(S) - 1).$$

So by equation (4.4), we see

$$(4n - a(S))(a(S) - 1) \geq n(n - 1).$$

Solving this quadratic inequality for $a(S)$, and rounding up to the nearest integer gives the conclusion (4.1). Using $(1 - x)^{1/2} \leq 1 - (1/2)x$ gives conclusion (4.2). Solving the quadratic inequality for n and rounding down gives the conclusion (4.3). \square

Values of $a(S)$ and the corresponding bounds on n and the approval ratio derived from equation (4.3) are given in Table 1.

$a(S)$	n	Approval Ratio	Observed n	Observed Approval Ratio
2	≤ 4	≥ 0.500	4	0.500
3	≤ 8	≥ 0.375	8	0.375
4	≤ 12	≥ 0.333	12	0.333
5	≤ 15	≥ 0.333	15	0.333
6	≤ 19	≥ 0.316	18	0.333
7	≤ 23	≥ 0.304	21	0.333
8	≤ 26	≥ 0.308	24	0.333
9	≤ 30	≥ 0.300	27	0.333
10	≤ 34	≥ 0.294	30	0.333
11	≤ 38	≥ 0.289	33	0.333
12	≤ 41	≥ 0.293	36	0.333

TABLE 1. On the left, this table shows for a given approval number the largest n that is given by inequality (4.3) as well as the resulting bound on the approval ratio derived from inequality (4.1). On the right, this table shows, for a given approval number, known examples of the largest n that has this approval number and the observed approval ratio in that case, obtained by a modification of a double- n string construction.

5. Modifying double- n string societies

In this section we give an example of a double-interval society with an approval ratio lower than the bound given by Theorem 3.6, thus showing that double- n strings do not always provide examples of societies with minimal approval ratios. We will require a new notation, called the *endpoint representation* of a society. We will encode a society as a sequence of symbols (corresponding to the approval sets) representing the order of the endpoints of all the approval sets, each prefixed by a “+” or a “-” to denote a left or right endpoint respectively. For example, the society in Figure 1 is represented as

$$+A + C + B - A + D - C + A - B - D + C + B - C + D - B - D - A.$$

PROPOSITION 5.1. *There exists a society of size $n = 8$ with approval number 3. Hence there exist n for which double- n strings do not produce the lowest possible approval numbers.*

The society shown in Figure 5 provides such a society. This example was derived from the double-8 string

$$ABCDEFGHIHEADFCGBH.$$

If each interval in the string overlaps two intervals on each side, this arrangement is missing the adjacencies AG, BE, BF, CA, DG, DG and CH and has duplicate adjacencies $BC, CD, DC, DE, DF, EG, FG,$ and GH . By doing a series of moves that interchanges endpoints in such a way as to introduce missing adjacencies (at the expense of duplicate adjacencies) without increasing the approval number, we

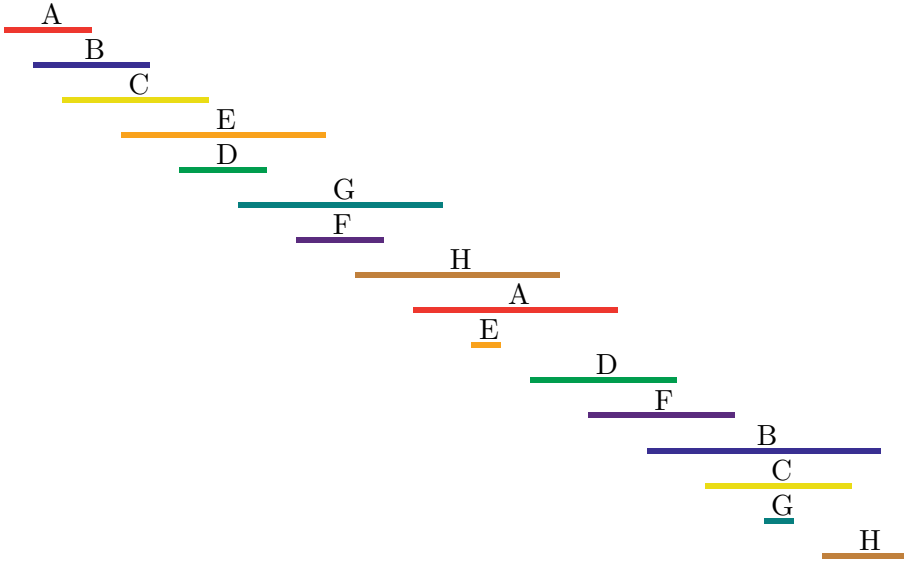


FIGURE 5. A society of size 8 with approval number 3.

arrive at the society

$$\begin{aligned}
 &+A + B + C - A + E - B + D - C + G - D + F - E + H - F + A - G \\
 &+E - E + D - H + F - A + B - D + C - F + G - G + H - C - B - H.
 \end{aligned}$$

We note that an example like this with $n = 8$ and $a(S) = 3$ cannot be achieved by a double- n string since the first symbol in a double- n string with diameter d is adjacent to at most $3d$ other symbols. Thus, as in Lemma 3.2, we have $n \leq 3d + 1 = 3(a(S) - 1) + 1$, and so the approval number of a double-8 string must be at least 4.

It is not clear how to systematically interchange endpoints to achieve all possible adjacencies. However, a “hill-climbing” algorithm which aims at making “smart” swaps produced societies with approval ratios given in Table 1. A description of the algorithm can be found in [4]. The results of the algorithm in Figure 6 suggest that the asymptotic approval ratio should be $1/3$.

6. Conclusion and Open Questions

We have studied pairwise-intersecting double-interval societies, and determined bounds for the minimum guaranteed approval ratio for such societies. Such questions naturally motivated the study of double- n strings, which represent certain special double-interval societies with low approval ratios. Although these do not necessarily provide the smallest such ratios, all of the known examples that provide smaller ratios come from modifying the double- n string construction.

There are numerous open questions.

- For double- n strings, is there a systematic way to construct strings of the smallest diameter?

$a(S)=3, n=8, AR=0.375:$ +A+B+F-F+G-A+F-B+C-C+D-G+E-D+H-F+G-G+A-E+D-H
 +C-A+B-D+E-E+H-H-B-C
 $a(S)=4, n=12, AR=0.333:$ +A+B+C+F-C+H-B+L-L+G-G+I-F+D-A+C-H+L-D+E-E+J
 -I+G-J+K-C+B-L+I-I+D-K+E-G+J-B+F-D+K-F+A-A+H
 -K-E-H-J
 $a(S)=5, n=15, AR=0.333:$ +A+B+C+D+E-C+G-A+O-G+F-D+K-F+J-J+N-E+C-B+A-O
 +D-K+H-H+M-N+J-M+L-L+I-D+F-A+G-C+N-I+L-N+H-J
 +M-F+K-G+I-K+B-B+E-E+O-H-M-I-L-O
 $a(S)=6, n=18, AR=0.333:$ +A+B+C+D+E+G-A+M-C+K-G+Q-Q+P-P+O-O+J-D+F-E+A
 -B+C-F+P-K+I-I+R-M+O-R+H-H+L-J+Q-L+N-A+G-C+J
 -J+F-P+I-O+R-N+H-Q+L-G+D-F+N-D+K-K+B-B+E-R+M
 -N-M-E-H-L-I
 $a(S)=7, n=21, AR=0.333:$ +A+B+C+D+E+F+I-A+K-K+N-N+R-R+G-E+T-B+J-T+U-G
 +Q-D+B-I+P-Q+M-C+H-F+L-J+S-U+O-B+D-D+C-C+I-I
 +F-F+G-P+Q-M+R-L+J-S+N-O+K-H+A-G+E-J+T-Q+P-P
 +M-M+U-R+S-S+O-U+L-N+H-T-H-K-L-E-O-A
 $a(S)=8, n=24, AR=0.333:$ +A+B+C+D+E+F+G+L-G+N-C+O-F+Q-O+M-N+T-B+H-H+K
 -Q+I-I+U-U+X-D+J-L+F-A+O-E+C-J+G-M+P-T+U-P+H
 -K+W-X+I-W+R-R+V-V+S-O+Q-C+J-F+N-G+B-U+X-S+P
 -H+R-I+V-X+W-Q+K-B+D-J+T-N+S-D+L-T+M-K+A-L+E
 -P-A-W-S-E-M-V-R

FIGURE 6. Output pairwise-intersection double-interval societies with given sizes and approval ratios found by a heuristic algorithm. Here AR denotes the approval ratio.

- Beyond double- n strings, is there a better general construction that yields societies with the lowest approval ratios?
- With double- n strings, we currently have Δ bounded by $0.348 \leq \Delta \leq 0.385$. Can we tighten the bounds on Δ ?
- What results can be obtained for triple-interval societies?
- What about higher-dimensional approval sets? What can be said if each voter's approval set consists of two convex sets in the plane?

Finally, we end with our initial conjecture, which now has more evidence as support.

CONJECTURE. *For all pairwise-intersecting double-interval societies S , the approval ratio*

$$\frac{a(S)}{n} \geq \frac{1}{3}.$$

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OFFICE OF THE PRESIDENT, HARVEY MUDD COLLEGE
E-mail address: klawe@hmc.edu

DEPARTMENT OF MATHEMATICS, WILLAMETTE UNIVERSITY
E-mail address: knyman@willamette.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY
E-mail address: jnscott@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, HARVEY MUDD COLLEGE
E-mail address: su@math.hmc.edu

Voting for committees in agreeable societies

Matt Davis, Michael E. Orrison, and Francis Edward Su

ABSTRACT. We examine the following voting situation. A committee of k people is to be formed from a pool of n candidates. The voters selecting the committee will submit a list of j candidates that they would prefer to be on the committee. We assume that $j \leq k < n$. For a chosen committee, a given voter is said to be satisfied by that committee if her submitted list of j candidates is a subset of that committee. We examine how popular is a most popular committee. In particular, we show there is always a committee that satisfies a certain fraction of the voters and examine what characteristics of the voter data will increase that fraction.

1. Introduction

The goal of this article is to examine the following voting situation: from a pool of n candidates, a committee of size k is to be formed. Every voter will submit an unordered list of j candidates that they would prefer be on the committee. We will assume throughout that $j \leq k < n$.

Once this voter data is collected, a committee will be chosen via some procedure. One way to select a winner of the election is to look for a most “popular” committee: a committee with the highest proportion of voter “approval”. (In case of ties, there may be more than one most popular committee.) We must make this notion precise.

Let us say that a voter *approves* a committee if it contains the list of j candidates that the voter submitted. Note that a voter may approve several committees, since several different k -element subsets will contain a given j -element subset if $j < k$.

We wish to answer questions regarding whether it is always possible, assuming certain conditions on the voters’ preferences, to find a committee approved by a certain proportion of voters. Phrased another way, we can ask how popular is a most popular (most-approved) committee? Can we make any minimal guarantees for the approval proportion of a most popular committee, given some geometric condition on the voter preferences?

Such questions have been addressed in voting contexts where each person is choosing just one candidate. Berg et al. [1] initiated the study of *agreeability conditions* for voting preferences over a linear political spectrum. In this situation, candidates are chosen from a line of possibilities, and each person is allowed to

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specify an interval of candidates that they find acceptable. In other words, their *approval set* is an interval in the real line. They call a society *super-agreeable* if every pair of voters has a mutually agreed candidate (i.e., their approval sets overlap) and they also consider other *agreeability* conditions that specify ways in which voter preferences, when taken in small subsets, are locally similar. They show how these produce global conclusions about how popular a most popular candidate must be, and give minimal guarantees for how many approval sets a most-approved candidate must lie in.

These ideas have been generalized to other kinds of geometric spaces that can be viewed as political spectra—circles [10], trees [8], multi-dimensional spaces [5]—as well as other kinds of approval sets [12] and agreeability conditions [4].

This paper considers similar questions in a new context: voting for committees. (See [2, 6, 7, 9, 11, 13, 14] for several different approaches to the study of voting for committees.) The set of n candidates is finite, and not viewed as a geometric space. However, voters are now allowed to specify a list of size j , and there is a geometric notion of how close two lists may be. The resulting geometric space of lists is called a *Johnson graph* $J(n, j)$ where each point is a j -element subset of an n -element set (see [3] for more on Johnson graphs).

Unlike the context of Berg et al. [1], here the space in which preferences are expressed is *different* from the space of potential outcomes of an election, which are the k -element subsets of the candidates, and which can be thought of in terms of the graph $J(n, k)$. In particular, each voter’s list in $J(n, j)$ produces an approval set in $J(n, k)$ consisting of all committees that contain the list. In this paper, we ask: what is the minimal guarantee for the popularity of a most popular committee in $J(n, k)$, and if we place conditions on how similar, or “agreeable”, the submitted lists are in $J(n, j)$, how does that change the guarantee?

We first demonstrate, in Theorem 2.2, a sharp lower bound for any voter distribution. We then show, in Theorem 5.1, that if all the votes lie within a “ball” in the space of lists, then the bound we can guarantee improves. We conclude with some extensions and open questions.

2. Definitions

Let us identify the candidates with the elements of the set $[n] = \{1, 2, \dots, n\}$. Any j -element subset of $[n]$ will be called a *list*. (Note that a list here is just a set, and is unordered.) Any k -element subset of $[n]$ will be called a *committee*.

We choose to think of given voter data as a probability distribution P on the set of lists, which we call the *voter distribution*. In particular, this distribution will specify for each list ℓ the *voting proportion* $P(\ell)$ —this is the fraction of voters who submitted the list ℓ . We shall use the same notation to describe the probability of a collection of lists, so that $P(\mathcal{A})$ describes the likelihood that a voter chose one of the lists in the collection \mathcal{A} .

If C is a committee, let

$$\pi_P(C) = \sum_{\ell \subseteq C, |\ell|=j} P(\ell)$$

denote the *approval proportion* of committee C with respect to voter data P : this is the fraction of voters that approve a given committee if the voter distribution on the set of all lists is P . We will write simply $\pi(C)$ if P is understood.

EXAMPLE 2.1. Assume that candidates 1 through 7 are being considered for a 4-person committee, and voters are asked to submit lists of three candidates. Define P so that $P(\{1, 2, 3\}) = 7/15$, while $P(\ell) = 2/15$ for the lists $\{4, 5, 6\}$, $\{4, 5, 7\}$, $\{4, 6, 7\}$, and $\{5, 6, 7\}$. (Then $P(\ell) = 0$ for all other lists.) For this choice of P , the list $\{1, 2, 3\}$ has the highest voting proportion, but by inspection, we see that $\pi_P(\{4, 5, 6, 7\}) = 8/15$, which is a higher approval proportion than any other committee.

This example shows that a most popular committee is not necessarily populated by candidates who are most popular. Even though each of candidates 1 through 3 appears in $7/15$ of the submitted lists, and candidates 4 through 7 each appear in only $6/15$ of the submitted lists, the committee $\{4, 5, 6, 7\}$ is still the most popular.

Our first observation is straightforward; it gives a minimum popularity for a most popular committee. The proof also illustrates a technique that we will exploit again later—that a most popular committee has at least as large an approval proportion than the *average* approval proportion over all committees.

THEOREM 2.2. *For any given voter distribution P , there exists a committee \widehat{C} with approval proportion satisfying*

$$\pi(\widehat{C}) \geq \frac{\binom{k}{j}}{\binom{n}{j}}.$$

PROOF. If we sum the approval proportions over all $\binom{n}{k}$ possible committees for C , we obtain

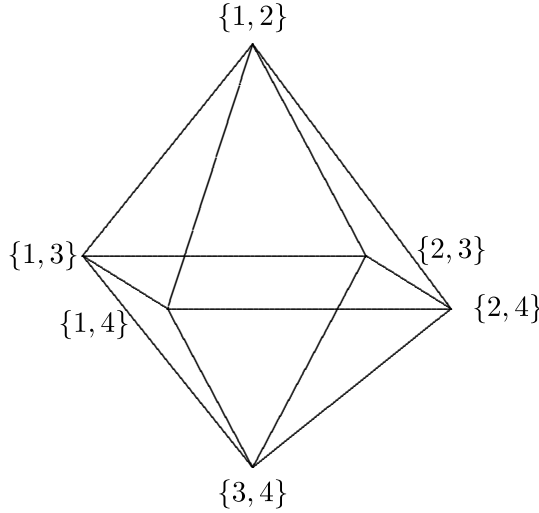
$$\begin{aligned} \sum_{C \subseteq [n], |C|=k} \pi(C) &= \sum_{C \subseteq [n], |C|=k} \left(\sum_{\ell \subseteq C, |\ell|=j} P(\ell) \right) \\ &= \binom{n-j}{k-j} \sum_{\ell \subseteq [n], |\ell|=j} P(\ell) \\ &= \binom{n-j}{k-j}. \end{aligned}$$

The first equality follows from the definition of the approval proportion. The second equality follows by noting that each list ℓ will be satisfied by the $\binom{n-j}{k-j}$ committees that include ℓ as a subset; hence the term $P(\ell)$ will appear $\binom{n-j}{k-j}$ times if we sum over lists, then committees. And the third equality is a consequence of P being a probability distribution.

Since there are $\binom{n}{k}$ terms in the sum, at least one committee \widehat{C} must have approval proportion $\pi(\widehat{C})$ at least as large as the average:

$$\pi(\widehat{C}) \geq \frac{\binom{n-j}{k-j}}{\binom{n}{k}} = \frac{(n-j)!k!}{(k-j)!n!} = \frac{\binom{k}{j}}{\binom{n}{j}}. \quad \square$$

Notice that this bound is the best possible, since it is achieved by, for example, the uniform distribution on the set of all lists.

FIGURE 1. The Johnson graph $J(4, 2)$.

3. Votes Within a Ball

We now investigate what characteristics of the voter distribution might improve the popularity of a most approved committee. For instance, in Theorem 2.2, the worst possible case is one in which the distribution is “spread out” over all possible lists uniformly. But if the voters are “agreeable” in some sense then we might be able to guarantee a committee that will be approved by more voters. As an example, if we knew that all of the voters’ lists were “close”, then this would suggest that the approval proportion of a most popular committee should be higher than the bound in Theorem 2.2. How can we describe the “closeness” of votes?

Consider the *Johnson graph* $J(n, j)$, whose vertices are the j -sets (i.e., j -element subsets) of $[n]$, with two j -sets v and w adjacent if they have exactly $j - 1$ elements in common. In our context, we think of the Johnson graph as the space of possible lists, with two lists adjacent if they differ by exactly one candidate. Then note that the graph distance d in $J(n, j)$ has a nice interpretation: for two lists v and w , the distance $d(v, w) = m$ if and only if $|v \cap w| = j - m$, i.e., v and w differ in exactly m places. This graph has diameter $D = \min(j, n - j)$, which is achieved when j -sets are as disjoint as possible.

We will explore how to improve the bound in Theorem 2.2 if all the submitted lists are within some “ball” in the Johnson graph about a central fixed list v . Given a non-negative integer r , we define the *ball of radius r around v* to be

$$B_r(v) = \{w \text{ a vertex of } J(n, j) \mid d(v, w) \leq r\}.$$

Then $B_r(v)$ is the entire Johnson graph if and only if $r \geq D$. In what follows, we assume that $j > 1$ so that the Johnson graph has diameter at least 2, and neighborhoods of vertices can be proper subsets of the set of vertices.

Figure 1, though a small example, gives some helpful intuition concerning the structure of Johnson graphs. We will examine *rings* of lists in $J(n, j)$ that are

equidistant from v . More formally, define the *ring* $R_r(v)$ by

$$R_r(v) = \{x \in J(n, j) \mid d(x, v) = r\},$$

for $0 \leq r \leq D$, which may be thought of as the ring of radius r about the vertex v . We note that if $d(x, v) = r$, then the list x must have $j - r$ candidates in common with the list v , and r candidates different from v . Thus

$$(3.1) \quad |R_r(v)| = \binom{j}{j-r} \binom{n-j}{r}.$$

For example, in Figure 1, if $v = \{1, 2\}$, the rings are:

$$\begin{aligned} R_0(v) &= \{\{1, 2\}\}, \\ R_1(v) &= \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}, \\ R_2(v) &= \{\{3, 4\}\}. \end{aligned}$$

EXAMPLE 3.1. Let $n = 6$, $k = 4$, and $j = 3$. Theorem 2.2 implies that for any P , there will always be a committee \hat{C} satisfying $\pi(\hat{C}) \geq \frac{1}{5}$. Now let $v = \{1, 2, 3\}$ and assume that P is supported on the ball $B_1(v)$, with $p = P(v)$. Then the complementary probability is

$$1 - p = \sum_{w \in R_1(v)} P(w) = \sum_{i=4}^6 P(\{1, 2, i\}) + P(\{1, 3, i\}) + P(\{2, 3, i\}).$$

Hence, since one of the summands must be at least as large as the average,

$$P(\{1, 2, i\}) + P(\{1, 3, i\}) + P(\{2, 3, i\}) \geq \frac{1}{3}(1 - p),$$

for some i with $4 \leq i \leq 6$. Then for $C = \{1, 2, 3, i\}$, we have

$$(3.2) \quad \pi_P(C) = P(v) + P(\{1, 2, i\}) + P(\{1, 3, i\}) + P(\{2, 3, i\}) \geq p + \frac{1}{3}(1 - p) \geq \frac{1}{3}.$$

In fact, if $p = 0$ and P is distributed uniformly on $R_1(v)$, then the inequality in (3.2) is equality, and $\frac{1}{3}$ is the highest approval proportion for any committee. So in this case, with the added assumption that P is supported on $B_1(v)$, we can improve the lower bound on $\pi_P(\hat{C})$ to $1/3$. (This will also be a consequence of Theorem 5.1 below.)

Rings about a vertex v always follow the same pattern: as r increases from 0 to D , the rings increase in size monotonically for a time, then decrease monotonically for a time. In particular, we can show:

LEMMA 3.2. $|R_r(v)| \leq |R_{r+1}(v)|$ holds if and only if

$$r \leq \frac{nj - j^2 - 1}{n + 2}.$$

PROOF. A straightforward calculation using (3.1) shows that both conditions are equivalent to the condition $(j - r)(n - j - r) \geq (r + 1)(r + 1)$. \square

It is helpful to visualize the Johnson graph as points on a sphere, with the list v at the north pole, and the rings $R_r(v)$ drawn as latitude lines. (This visualization explains our terminology and captures the behavior of the relative sizes of the rings.)

Suppose the voter distribution P is supported on a neighborhood $B_\rho(v)$ of radius ρ . To find a minimal guaranteed approval proportion, we are interested

in the worst-case scenario – the distribution that leads to a worst possible “best” committee. Intuitively, we would expect this worst-case scenario to occur when the voters’ lists are spread out as far away from v as possible, i.e., on the ring $R_\rho(v)$. However, if ρ is too large, the outermost ring $R_\rho(v)$ will be too small, and votes on that ring will be too close together. In that case, the worst-case scenario will result from a more complicated distribution of voters.

EXAMPLE 3.3. Let $n = 6$, $k = 4$, and $j = 3$, and let $v = \{1, 2, 3\}$. In Example 3.1, we showed that if the voter distribution P is supported on $B_1(v)$, then there must be a committee C with $\pi_P(C) \geq 1/3$. As stated there, if P is in fact the uniform distribution on $R_1(v)$, then every committee C that contains all of v has $\pi_P(C) = 1/3$, and $1/3$ is the maximum approval proportion in this case. Moreover, if $P(v) > 0$, then the discussion in Example 3.1 shows that one of the committees that contains v will have an approval proportion strictly larger than $1/3$. Thus the “worst-case” scenario occurs only when P is supported on the outermost ring $R_1(v)$.

Conversely, if P were supported on $B_2(v)$, then the worst-case scenario cannot result from P being supported only on $R_2(v)$. Because $R_2(v) = R_1(\{4, 5, 6\})$, the previous example shows that at least one committee C would have $\pi_P(C) \geq 1/3$ if P were supported on $R_2(v)$. However, if P is the uniform distribution on $B_2(v)$, then it may be checked that the largest approval proportion for any committee is $4/19$. In general, we expect that the worst-case scenario will occur when P is supported on $R_\rho(v)$ only when ρ is small.

4. Concentric Voter Distributions

We begin by showing that the rings, aside from being helpful in visualizing the neighborhood $B_\rho(v)$, are also quite important to our analysis. We define a *concentric distribution centered at v* to be a distribution P such that, for any r and for any two lists ℓ_1 and ℓ_2 in $R_r(v)$, the proportion of voters choosing ℓ_1 and ℓ_2 are the same: $P(\ell_1) = P(\ell_2)$. That is, the votes for lists in a given ring are distributed uniformly on that ring.

Let w_r denote the *weight* of the ring $R_r(v)$: it is the sum of the voting proportions of elements of the ring $R_r(v)$. Then a concentric distribution is completely determined by the weights w_r , and these weights sum to 1.

LEMMA 4.1. *For any voting data P , let P° represent the concentric distribution centered at v that has the same weights as P . Then if C is a committee with the highest approval proportion $\pi_P(C)$, and C° is a committee with the highest approval proportion $\pi_{P^\circ}(C^\circ)$, then*

$$\pi_P(C) \geq \pi_{P^\circ}(C^\circ).$$

In other words, uniformizing the voter distribution concentrically over the rings centered at v can only decrease the popularity of a most popular committee.

PROOF. We will treat the weights as fixed for now, and partition the set of committees into classes by the number of elements a committee differs from the central list v . We’ll show that the approval proportion π_P dominates π_{P° in each class.

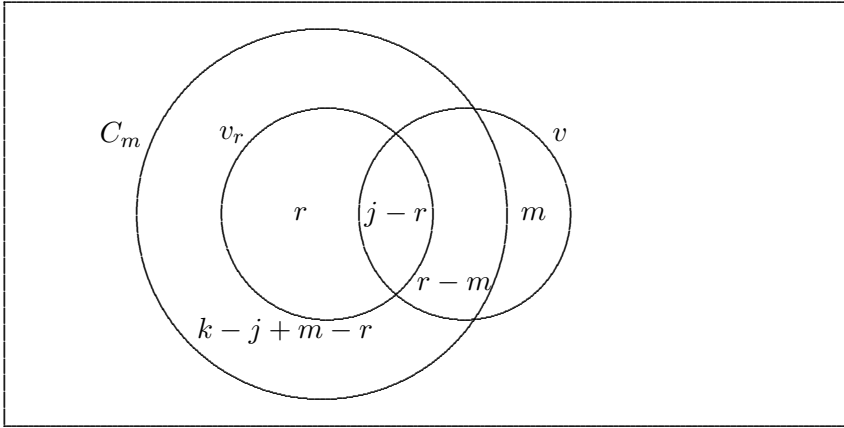


FIGURE 2. Committee C_m satisfies list v_r , which is distance r from list v .

Let \mathcal{C}_m denote the set of committees that differ from the list v in exactly m candidates, i.e., m candidates are “missing”. Thus

$$\mathcal{C}_m = \{C \subseteq [n] : |C| = k, |C \cap v| = j - m\}.$$

For instance, \mathcal{C}_0 consists of all committees that contain the list v .

Note that $m \leq n - k$, because any k -person committee in \mathcal{C}_m is missing $n - k$ candidates, including by construction exactly m candidates in v . We also see $m \leq j$, since v has only j candidates. Every committee belongs to exactly one \mathcal{C}_m for some m , and there are $\binom{j}{j-m} \binom{n-j}{k+m-j}$ committees in \mathcal{C}_m .

We can count the number of committees in \mathcal{C}_m that contain a fixed list v_r in the ring $R_r(v)$. A list $v_r \in R_r(v)$ that is contained in a committee $C \in \mathcal{C}_m$ can be visualized as in Figure 2, which is labeled with the sizes of the various subsets involved. To construct such a committee C , we must choose $r - m$ elements from the r elements in $v \setminus v_r$, and $k - j + m - r$ elements from the $n - j - r$ elements not in $v \cup v_r$. Thus there are $\binom{r}{r-m} \binom{n-j-r}{k-j+m-r}$ such committees.

Then consider:

$$\sum_{C \in \mathcal{C}_m} \pi_P(C) = \sum_{C \in \mathcal{C}_m} \sum_{\ell \subseteq C, |\ell|=j} P(\ell).$$

Note that $P(\ell)$ appears multiple times in this sum, once for every instance where a committee C in \mathcal{C}_m contains ℓ . The number of times that happens depends on what ring ℓ is in; for ℓ in $R_r(v)$, the number of committees in \mathcal{C}_m that contain ℓ is $\binom{r}{r-m} \binom{n-j-r}{k-j+m-r}$ as calculated above. So our sum $\sum_{C \in \mathcal{C}_m} \pi_P(C)$ now becomes:

$$\begin{aligned} \sum_{C \in \mathcal{C}_m} \pi_P(C) &= \sum_{r=m}^D \left[\sum_{\ell \in R_r(v)} P(\ell) \right] \binom{r}{r-m} \binom{n-j-r}{k+m-j-r} \\ &= \sum_{r=m}^D w_r \binom{r}{r-m} \binom{n-j-r}{k+m-j-r}, \end{aligned}$$

where w_r is the weight of the ring $R_r(v)$. The sum index starts at $r = m$ because all subsets of \mathcal{C}_m by construction must differ from v by at least m elements, and the index ends at D , the diameter of the graph.

Then at least one of the committees $\widehat{C} \in \mathcal{C}_m$ must have approval proportion equal to or better than average, or in other words,

$$(4.1) \quad \begin{aligned} \pi_P(\widehat{C}) &\geq \frac{1}{\binom{j}{m} \binom{n-j}{k+m-j}} \sum_{r=m}^D w_r \binom{r}{r-m} \binom{n-j-r}{k+m-j-r} \\ &= \sum_{r=m}^D w_r \frac{r!}{(r-m)!} \frac{(n-j-r)!}{(k+m-j-r)!} \frac{(j-m)!(k+m-j)!}{j!(n-j)!}. \end{aligned}$$

Now, consider a similar analysis using concentrically distributed voter data P° centered at v with the same ring weights w_r as P . Specifically, we assume that each list in $R_r(v)$ receives voting proportion

$$\frac{w_r}{\binom{j}{j-r} \binom{n-j}{r}}.$$

Then, we note that a committee C° in \mathcal{C}_m will contain $\binom{j-m}{j-r} \binom{k+m-j}{r}$ of the lists in the ring $R_r(v)$. Again referring to Figure 2, we see that a list that is in $R_r(v)$ must contain $j-r$ elements of v chosen from the $j-m$ such elements in C° , and must contain r candidates not in v chosen from the $k+m-j$ such candidates in C° .

Thus, if $C^\circ \in \mathcal{C}_m$, we have

$$(4.2) \quad \begin{aligned} \pi_{P^\circ}(C^\circ) &= \sum_{r=m}^D w_r \frac{\binom{j-m}{j-r} \binom{k+m-j}{r}}{\binom{j}{r} \binom{n-j}{r}} \\ &= \sum_{r=m}^D w_r \frac{(j-m)!}{(r-m)!} \frac{(k+m-j)!}{(k+m-j-r)!} \frac{r!(n-j-r)!}{j!(n-j)!}. \end{aligned}$$

So every committee in \mathcal{C}_m has the same approval proportion under P° . Comparing the final expressions in (4.1) and (4.2), we see that both are equal to

$$\sum_{r=m}^D w_r b_{r,m}$$

where

$$b_{r,m} = \frac{r!}{(r-m)!} \frac{(k+m-j)!}{(k+m-j-r)!} \frac{(j-m)!}{j!} \frac{(n-j-r)!}{(n-j)!}.$$

Thus

$$\pi_P(\widehat{C}) \geq \pi_{P^\circ}(C^\circ).$$

That is, the approval proportion of a most popular committee in \mathcal{C}_m under P will equal or exceed the approval proportion of any committee in \mathcal{C}_m under P° . Then, taking the maximum over all m , we obtain the desired result. \square

The numbers $b_{r,m}$ are integral to comparing the number of lists contained in various committees. Most important are their relative sizes for a fixed value of r .

LEMMA 4.2. *We have $r \leq j \left[1 - \frac{j-m}{k+1}\right]$ (using the notation of Lemma 4.1) if and only if $b_{r,m} \geq b_{r,m+1}$.*

As an example of this, if $m = 0$ and j is approximately half of k , then we obtain the condition that r is less than half of j .

PROOF. We start with the inequality $b_{r,m} \geq b_{r,m+1}$. After expanding the binomial coefficients and canceling common terms, we see this is equivalent to

$$(j - m)(k + m + 1 - j - r) \geq (r - m)(k + m + 1 - j)$$

which is equivalent to $r \leq \frac{j(k+1+m-j)}{k+1}$, as desired. \square

5. Main Theorem

We are now in a position to prove our main theorem. We now assume that the voter distribution P is supported in some ball $B_\rho(v)$ centered at a list v , with $\rho < D$ so that $B_\rho(v)$ is not the entire Johnson graph.

THEOREM 5.1. *Let $v = \{1, 2, \dots, j\}$ and consider a voter distribution P supported on $B_\rho(v)$ in the Johnson graph. If $\rho \leq j \left[1 - \frac{j}{k+1}\right]$, then there exists a committee that satisfies at least $\frac{\binom{k-j}{\rho}}{\binom{n-j}{\rho}}$ of the voters.*

PROOF. Since we are trying to minimize the proportion of voters satisfied by a most popular committee, Lemma 4.1 shows that we may as well assume the voting distribution is concentric, which we do from now on. Now, for each m , let C_m be any committee in \mathcal{C}_m . Then the expression given in (4.2) gives the proportion of voters satisfied by this committee in terms of the w_r . The maximum over all m is an optimal committee for the given profile, so our goal is to minimize that maximum value. But, under our assumptions, we see that

$$r \leq \rho \leq j \left[1 - \frac{j}{k+1}\right] \leq \frac{j(k+1+m-j)}{k+1}.$$

Then Lemma 4.2 shows that $b_{r,m} \geq b_{r,m+1}$. Thus, under our assumptions for any fixed values of w_r , we always have

$$\sum_{r=m}^{\rho} w_r b_{r,m} \geq \sum_{r=m}^{\rho} w_r b_{r,m+1},$$

since the weights w_r are non-negative. Thus we see that C_m is more widely approved of than C_{m+1} , and C_0 is always a most-approved committee.

Then we have to choose the w_r to minimize the value of $\sum_{r=0}^{\rho} w_r b_{r,0}$. But

$$\begin{aligned} b_{r,0} &= \frac{(k-j)!}{(k-j-r)!} \frac{(n-j-r)!}{(n-j)!} \\ &= \frac{(k-j)!}{(n-j)!} \cdot (n-j-r)(n-j-r-1) \dots (k-j-r+1). \end{aligned}$$

But there are always $n-k$ terms in $(n-j-r)(n-j-r-1) \dots (k-j-r+1)$, beginning at $n-j-r$. Thus, $b_{0,0} \geq b_{1,0} \geq \dots$, and the minimum value of $\pi(C_0) = \sum_{r=0}^{\rho} w_r b_{r,0}$ is achieved by letting $w_\rho = 1$, and letting every other w_r be 0. In this case, the expression in (4.2) gives the desired lower bound. \square

6. Extensions

There are some natural extensions of Theorem 5.1 to other situations that may be of interest. The first concerns what we can guarantee if only a fraction of the voters submit votes in a neighborhood $B_\rho(v)$.

COROLLARY 6.1. *Let $\rho \leq j \left[1 - \frac{j}{k+1}\right]$. Suppose P is a voter distribution in which a proportion α of the voters select lists in a ball $B_\rho(v)$, e.g., $P(B_\rho(v)) = \alpha$. Then there exists a committee C with approval proportion*

$$\pi_P(C) \geq \frac{\binom{k-j}{\rho}}{\binom{n-j}{\rho}} \cdot \alpha.$$

PROOF. This follows from applying Theorem 5.1 to just those voters whose votes lie in $B_\rho(v)$. We see there exists a committee that satisfies $\frac{\binom{k-j}{\rho}}{\binom{n-j}{\rho}}$ of those voters, which make up $\frac{\binom{k-j}{\rho}}{\binom{n-j}{\rho}} \cdot \alpha$ of the entire set of voters. \square

We can also consider slightly more general election procedures. Suppose in choosing a k -member committee from a pool of n candidates, that we allow voters to submit a smaller unranked list between 1 and j candidates that they would prefer to have on the committee. If these votes are sufficiently similar to v in a sense that we make precise below, then we can obtain a result like (and because of) Theorem 5.1.

COROLLARY 6.2. *Consider a distribution of votes over lists of size less than or equal to j , in such a way that each submitted list is a subset of some list in a ball $B_\rho(v)$ of radius $\rho \leq j \left[1 - \frac{j}{k+1}\right]$ in the Johnson graph. Then there exists a committee that satisfies at least $\frac{\binom{k-j}{\rho}}{\binom{n-j}{\rho}}$ of the voters.*

PROOF. We can convert such a distribution on subsets of size $\leq j$ into one on j -sets satisfying the assumptions in Theorem 5.1, in a way that can only decrease the number of satisfied voters. To see this, consider any voter who submits a list v containing fewer than j candidates. We replace that list with a list v' in $B_\rho(v)$ that contains v as a subset. (Note that a voter who submitted v' would be satisfied by strictly fewer committees than a voter who submitted the list v .) Then Theorem 5.1 will apply to the resulting voting distribution, and because our alterations might only have decreased the number of satisfied voters, the lower bound still holds. \square

One may also consider elections using *thresholds* as described in [7], and ask whether our methods would extend in that context, in which a voter submitting a list ℓ approves a committee C if C contains sufficiently many members of ℓ . Our main result is the special case that each voter only approves committees that contain *all* j of the candidates from their list. If voters approve committees that contain at least s of the candidates from their list, for some s between 0 and j , then the analogous result to Lemma 4.1 holds and can be proved in a similar way. However, the analogues of the numbers $b_{r,m}$ are significantly more complicated, making the development of a result similar to Lemma 4.2 a barrier to proving a generalization of the main theorem.

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DEPARTMENT OF MATHEMATICS, MUSKINGUM UNIVERSITY, NEW CONCORD, OHIO 43762
E-mail address: mattd@muskingum.edu

DEPARTMENT OF MATHEMATICS, HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA 91711
E-mail address: orrison@math.hmc.edu

DEPARTMENT OF MATHEMATICS, HARVEY MUDD COLLEGE, CLAREMONT, CALIFORNIA 91711
E-mail address: su@math.hmc.edu

Selecting diverse committees with candidates from multiple categories

Thomas C. Ratliff

ABSTRACT. When selecting committees, the voters often place priority on the overall composition of the committee in such a way that their preferences cannot be decomposed into preferences on individual candidates. This is frequently reflected in the voters' desire that the elected committee respect some diversity criterion, such as gender or academic rank. One aspect that makes committee elections unwieldy is that it is usually impractical to ask voters for a complete ranking of all possible committees. Our approach is to ask voters for their top-ranked committee and devise weighted voting methods that respect the diversity criterion. We expand previous results when the diversity criterion consists of two categories to the analysis of three or more categories.

1. Introduction

The problem of designing elections to select a committee can raise different issues than those of standard voting situations that produce a single winner or a ranking of all candidates. One example of this difference is when the voters expect the committee to contain members from several different categories. We can easily imagine circumstances where a particular diversity criterion may be important: The voters may want to ensure that a faculty committee contains members at the ranks of assistant, associate, and full professor; an organization may require that the speakers on a panel represent a variety of viewpoints; or a community may want to select representatives from each of five precincts in the community for a task force.

Rather than being of purely abstract interest, there are examples of such considerations that have occurred in practice. Ratliff [12] describes an election at Wheaton College where the desire was to have both women and men on a search committee, yet only men were elected, and Ratliff and Saari [13] explain how elections to one class of the US National Academy of Sciences should contain members from four different sections. Others have explored the barriers to diverse representation in legislatures with single-member districts, including Fréchet, Maniquet, and Morelli [8] who discuss the impact of a gender parity law in the French Assembly.

One view of committee elections is consistent with the idea of proportional representation as expressed by Chamberlin and Courant [6] where each voter has a representative on the deliberative body to reflect their views and interests. A

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consequence of defining this strong connection between the voter and a single representative is that it is reasonable to expect voters to express their preferences by providing a ranking of the individual candidates. This approach is common in the literature, including the paper by Chamberlin and Courant [6] and papers by Gehrlein [9], Barberà, Sonnenschein, and Zhou [2] and Barberà, Massó, and Neme [3].

There are other possible assumptions about the voters' preferences in committee elections. Bock, Day, and McMorris [4] propose a method that was motivated by a consensus approach used in biological sequence analysis where each voter selects a single candidate and the size of the committee is determined after the votes are cast. Fishburn [7] assumes that each voter has an ordering of all possible committees subject to certain conditions, one of which yields an asymmetric weak order on the set of candidates for each voter. Brams, Kilgour, and Sanver [5] propose a method where voters mark all candidates that are acceptable to them as members of the committee, and the set of candidates that minimizes the maximum Hamming distance to the individual voters is chosen.

Unlike these approaches, we are interested in committee elections where the voters have a preference for the composition of the entire selected group and the preferences may not be separable into preferences on individual candidates. A second Wheaton election described in Ratliff [12] demonstrates that this situation may occur in practice. The voters ranked the eight possible committees, and for over half of the voters, their top-ranked and bottom-ranked committees had at least one candidate in common. Thus, for at least half of the voters, their preferences are not separable into preferences on individual candidates. Notice that non-separable preferences can be a reflection of the voters' desire that the committee achieve a particular diversity criterion.

There are more details in Ratliff [12] and Ratliff and Saari [13], but we can see how the problem of selecting a diverse committee can arise with a simple example. Suppose an academic department is electing a committee of size $k = 3$ where the candidates consist of three tenured faculty, Tanya, Terry, and Ted, and three untenured faculty, Uri, Ursula, and Uma. We have 12 voters, each of whom selects a diverse committee consisting of both tenured and untenured members.

EXAMPLE 1.

# Voters	Tanya	Terry	Ted	Uri	Ursula	Uma
3	x				x	x
3		x		x		x
5			x	x	x	
1	x	x			x	
Totals	4	4	5	8	9	6

Despite every voter's preference for a diverse committee, plurality selects the homogeneous committee {Uri, Ursula, Uma}. One solution is to treat each possible committee as an indivisible unit and then apply one of the standard voting approaches, such as the Borda count or an approval voting ballot. However, the number of possible committees grows very large for even small committee sizes, making it difficult for voters to provide meaningful preferences. In our simple example, there are 18 possible diverse committees.

The approach in this paper is to find a middle ground that does not require the voters to rank all possible committees, but instead only asks for their top-ranked committee. Continuing the approach in Ratliff and Saari [13], we will consider methods that assign weights to the individual candidates and determine when we are guaranteed to meet the universally desired diversity criterion. Since we are not insisting upon a full ranking of all committees, there will necessarily be some tradeoffs. The good news is that we obtain positive results, but the required compromises may be so restrictive when there are three or more diversity categories as to give pause before using this approach.

The paper is organized as follows. Section 2 gives the basic framework and previous results from Ratliff and Saari [13] when there are two candidates for each position on the committee, each coming from one of two diversity categories. Section 3 describes the case of two categories without candidates slotted into specific positions, and Section 4 builds on these techniques to describe the situation of three or more diversity categories.

2. Basic framework

For the remainder of the paper, we make the following assumptions about the voters' preferences and the voting method used to select the committee of size k .

- There is a universally recognized diversity criterion that every voter desires to see reflected in the composition of the committee. This criterion may divide the candidates in two categories (e.g. tenured and untenured faculty) or three or more categories.
- Each voter specifies their top-ranked committee of size k . Since the diversity criterion is assumed to be universal, every voter's top-ranked committee includes at least one candidate from each diversity category.
- We will consider voting methods that assign weights to the candidates, and the winning committee consists of the k candidates with the largest tally.
- We assume that our voting method is neutral with respect to the categories and candidates. We will explain this requirement in more detail below.

The simplest method that falls within our framework is straight plurality: Assign one point to each candidate in a voter's preference, and the top k candidates form the winning committee. As we have seen in Example 1, this can lead to a homogeneous committee. It will be convenient in the exposition to denote the tenured faculty by $\{t_1, t_2, t_3\}$ and the untenured faculty by $\{u_1, u_2, u_3\}$. The election now becomes

EXAMPLE 2.

# Voters	t_1	t_2	t_3	u_1	u_2	u_3
3	x				x	x
3		x		x		x
5			x	x	x	
1	x	x			x	
Totals	4	4	5	8	9	6

The goal of this paper is to categorize when we can guarantee the outcome reflects the preference for a diverse committee by assigning values other than the uniform weights used by plurality.

To illustrate the neutrality condition, suppose a voting method assigns the weights $(1, 3, 2)$ to the committee (t_1, t_2, u_2) . Then, the method must also allow the allocation $(1, 3, 2)$ to (t_2, t_3, u_1) , (u_1, u_2, t_1) , (t_2, t_1, u_2) or any committee where the first two candidates are from the same category and the third is the “diversity” candidate from the other category. Notice that in this terminology we are not specifying all candidates from one category as the diversity candidates, but rather the candidate from the category that is underrepresented in a voter’s ballot is the diversity candidate for that ballot. For example, u_1 is the diversity candidate in (t_2, t_3, u_1) but t_1 is the diversity candidate in (u_1, u_2, t_1) .

2.1. Previous results. Ratliff and Saari [13] considers the case where the diversity criterion consists of two categories and there are two candidates, one from each category, slotted for each of the k positions on the committee. Example 2 meets this requirement where we have three slots consisting of

$$(t_1 \text{ vs } u_1, \quad t_2 \text{ vs } u_2, \quad t_3 \text{ vs } u_3)$$

Then plurality gives us $(u_1 > t_1, u_2 > t_2, u_3 > t_3)$ and the homogeneous outcome of (u_1, u_2, u_3) . However, we can guarantee a diverse outcome by a careful selection of the weights.

THEOREM 1. [13, Theorem 4] *In electing a k -person committee from among candidates who are slotted in k divisions, suppose each division has two candidates representing two different categories. To reflect a universal intent shared by all voters to elect a committee with representatives from each of these two categories, an admissible ballot must have at least one candidate from each category. The diversity objective always can be achieved if and only if individual weights assigned to diversity candidates are greater than or equal to the weights assigned to non-diversity candidates and the sum of weights assigned to the candidates of each category are equal.*

To apply this theorem to Example 2, we could use a total weight of 2 per category and give 1 point to each candidate from the same category and 2 points to the diversity candidate from the other category. For example, the preference (t_1, u_2, u_3) would assign weights $(2, 1, 1)$. Then our tallies become:

EXAMPLE 3.

# Voters	t_1	t_2	t_3	u_1	u_2	u_3
3	2				1	1
3		2		1		1
5			2	1	1	
1	1	1			2	
Totals	7	7	10	8	10	6

giving $(u_1 > t_1, u_2 > t_2, t_3 > u_3)$ and the diverse outcome of (u_1, u_2, t_3) .

A similar process holds for larger committees. For example, with a committee of size $k = 5$, we could use a total weight of 12. Then the preference $(t_1, u_2, t_3, u_4, u_5)$ would be given weights $(6, 4, 6, 4, 4)$, and the preference $(t_1, u_2, u_3, u_4, u_5)$ would be given weights $(12, 3, 3, 3, 3)$. Notice that all candidates in the same category are assigned the same weight, but this is not a requirement of the theorem.

An obvious question is how worried we should be about homogeneous outcomes using the plurality. This will depend on our assumptions about the underlying probability model. Two of the more popular models are the Impartial Anonymous

Culture (IAC), where each profile is considered to be equally likely to appear, and the Impartial Culture (IC), where each voter is equally likely to choose each committee. By using geometric methods, Ratliff and Saari [13, Theorem 1] determines the probability of obtaining a homogeneous outcome for $k = 3$. Assuming IAC, the probability of obtaining a homogeneous outcome approaches 0.0625 as the number of voters approaches infinity, and assuming IC, the probability of obtaining a homogeneous outcome approaches 0.0877 as the number of voters approaches infinity. In other words, we should expect a homogeneous outcome in approximately one out of every 16 elections assuming IAC and in approximately one out of every 12 elections assuming IC.

2.2. Ehrhart polynomials. We can repeat, and extend, the probability calculations for IAC by the use of Ehrhart polynomials. See Lepelley [11] for a very nice introduction to this theory applied to voting. We will briefly recap the major points in the context of determining the probability of a homogeneous outcome for $k = 3$ slotted elections using plurality. There are six possible diverse committees, and we can denote the number of voters with each preference by n_1, n_2, \dots, n_6 . We can express the conditions that give the homogeneous outcome $\{u_1, u_2, u_3\}$ as a system of linear inequalities in n_1, \dots, n_6 . Further, if we restrict to a fixed number of voters, $n_1 + \dots + n_6 = n$, then the system defines a polytope in \mathbb{R}^5 . If we can determine the number of integer lattice points in the polytope, then we will know the number of elections with n voters that will give a homogeneous outcome. We can then use the well-known fact that the number of elections with six options is given by $\binom{n+5}{5} = \frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{5!}$ to calculate the probability of a homogeneous outcome with n voters. We can then find the long range behavior by taking the limit as n approaches infinity.

Ehrhart polynomials provide a tool to count the number of integer lattice points in the dilation of a polytope with rational vertices, where the dilation of a polytope P is the set $nP = \{n\mathbf{x} \mid \mathbf{x} \in P\}$. Thus, if we define our original polytope P by $n_1 + \dots + n_6 = 1$, then the number of integer lattice points in nP will give the number of elections with n voters that determine a homogeneous outcome. If the vertices of the polytope P are integers, then there is a single Ehrhart polynomial f associated with P . If the vertices are non-integer rationals, then we obtain a list of polynomials g_1, \dots, g_p where p is a divisor of the lcm of the denominators of the vertices of P . In this case, the Ehrhart polynomial is a *quasi-polynomial* f of degree d and period p where $f(n) = g_i(n)$ if $n \equiv i \pmod{p}$. In either case, $f(n)$ gives the number of integer lattice points in the dilation nP . An important feature of the Ehrhart polynomial is that the leading coefficient is independent of n , and thus is the same for all g_i .

Fortunately, there are tools available to aid with the computation. The workflow for this paper was to use the the program *lrs* from Avis [1] to calculate the vertices of the polytope P . This gave an upper bound for the period of the quasi-polynomial. The program *count* from Köppe [10] has many capabilities, including the ability to calculate the value of the Ehrhart polynomial for a specific value of n . Since we know the degree and the period, these programs allow us to find enough specific values so that we can then determine the Ehrhart polynomial through interpolation. Using these techniques, we found that the Ehrhart polynomial for the

$k = 3$ slotted case is given by

$$f(n) = \frac{1}{3840}n^5 + \left[\frac{1}{128}, \frac{1}{256} \right] n^4 + \left[\frac{17}{192}, \frac{7}{384} \right] n^3 + \left[\frac{15}{32}, \frac{3}{128} \right] n^2 + \left[\frac{137}{120}, -\frac{71}{3840} \right] n + 1$$

where the notation $\left[\frac{1}{128}, \frac{1}{256} \right]$ indicates that the coefficient n^4 is $\frac{1}{128}$ if $n \equiv 0 \pmod{2}$ or is $\frac{1}{256}$ if $n \equiv 1 \pmod{2}$. Thus, the probability of achieving the homogeneous outcome $\{u_1, u_2, u_3\}$ with n voters is given by $\frac{f(n)}{\frac{1}{5!}(n+5)(n+4)(n+3)(n+2)(n+1)}$. Notice that both the numerator and denominator are polynomials of degree 5, and therefore the limit as n approaches infinity depends only on the coefficients of the largest terms.

$$\lim_{n \rightarrow \infty} \frac{f(n)}{\binom{n+5}{5}} = \frac{5!}{3840} = 0.03125$$

Since our situation is completely symmetric with respect to the categories, the probability of achieving the other homogeneous outcome $\{t_1, t_2, t_3\}$ is also 0.03125. Thus, the probability of obtaining a homogeneous outcome approaches 0.0625 as the number of voters approaches infinity, agreeing with the previous result in Ratliff and Saari [13].

The advantage of using Ehrhart polynomials for the calculations is that these techniques apply to other scenarios as well, whereas the geometric-based results from Ratliff and Saari [13] are specific to the $k = 3$ case. We will be able to take advantage of the fact that the final calculation depends only upon the leading coefficient to reduce the computations to finding a single polynomial rather than the entire quasi-polynomial. For a committee of size $k = 4$ with the candidates from two categories slotted against each other, there are 14 diverse committees and two homogeneous committees. Following the same process as above, we find that the Ehrhart quasi-polynomial corresponding to one homogeneous outcome is of degree 13 and period 12 with leading coefficient $\frac{53}{12752938598400}$. Thus, as the number of voters approaches infinity, the probability of obtaining a homogeneous outcome is

$$13! \cdot \frac{53}{12752938598400} \cdot 2 \approx 0.05176$$

Therefore, we should expect a homogeneous outcome using plurality in approximately one out of every 20 elections.

In theory, we can repeat this process for larger committees. However, the calculations are very computationally intensive. For $k = 5$, there are 30 diverse committees and two homogeneous committees leading to a Ehrhart quasi-polynomial of degree 29. On a moderately powered desktop computer, I halted the computation after approximately 40 hours with an indeterminate time remaining. A faster computer and an improvement in the algorithms may make these direct computations attainable in the future.

Although we do not know the exact probability for $k = 5$, there are good reasons to believe that the probability of a homogeneous outcome will decrease as k increases. The geometric intuition for this claim is explained in Section 3.6.

3. Two categories, no slots

There are two natural extensions of the previous results to consider. In this section we remove the restriction that the candidates from the two categories are slotted against each other, and we consider the case of the diversity criterion consisting of more than two categories in the following section. Our goals are fourfold:

We will show that plurality can give a homogeneous committee; we will show that the ability to guarantee a diverse outcome is sensitive to the number of candidates in each category; we will show when we can assign weights to guarantee a diverse outcome; and we will calculate the probability of a homogeneous outcome using plurality. The last goal is somewhat limited by computational restrictions, but we will also develop a geometric framework to give some intuition for what we should expect when the direct probability calculations are not feasible.

Assume that the diversity criterion divides the candidates into two categories, $\{t_1, t_2, \dots, t_r\}$ and $\{u_1, u_2, \dots, u_s\}$. Notice that we allow $r \neq s$ so there may be a different number of candidates in each group. As before, we are selecting a committee of size k , and we assume that every voter specifies a diverse committee on their ballot. Also note that $r + s > k$ or else the elections is impossible ($r + s < k$) or trivial ($r + s = k$). We also assume that $r > 1$. Otherwise, any diverse committee must contain t_1 , and the election reduces to selecting $k - 1$ candidates from $\{u_1, u_2, \dots, u_s\}$ with no diversity criterion. Similarly, we assume $s > 1$.

Our analysis breaks down into several cases as determined by the regions shown in Figure 1.

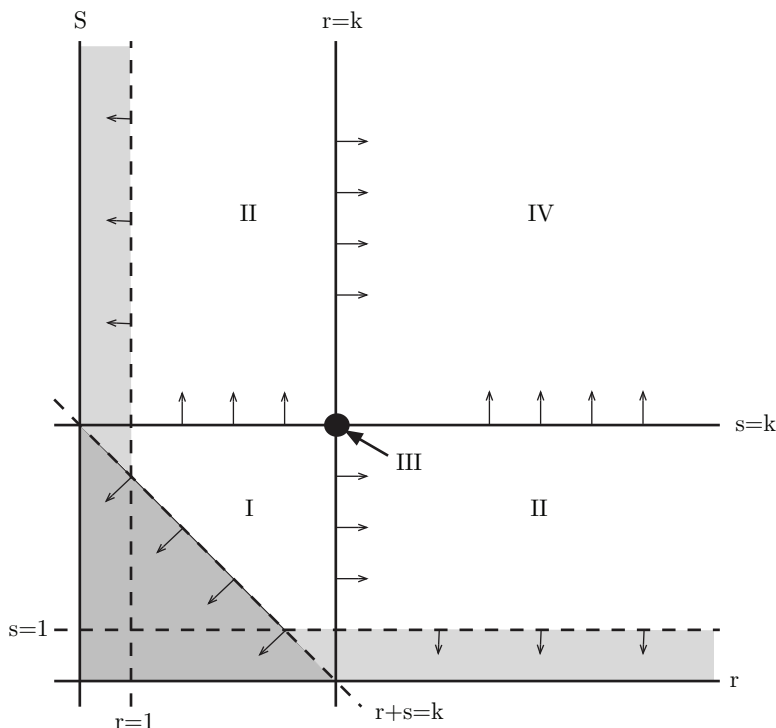


FIGURE 1. The fundamental regions for the two category case with candidates $\{t_1, t_2, \dots, t_r\}$ and $\{u_1, u_2, \dots, u_s\}$. The arrows indicate to which region the boundaries belong.

3.1. Case I: $r < k$, $s < k$, and $r + s > k$. We are guaranteed a diverse committee independent of the voting method since it is impossible to select a subset of k candidates without including at least one candidate from each category.

3.2. Case II: $r < k$ and $s \geq k$ (or $s < k$ and $r \geq k$). Using a similar construction to Example 2, we can show that it is possible for plurality to give a homogeneous committee. We will look at a specific case before formulating the general construction.

Suppose we are selecting a committee of size $k = 3$ with candidates $\{t_1, t_2\}$ and $\{u_1, u_2, u_3\}$. Consider the following election:

EXAMPLE 4.

# Voters	t_1	t_2	u_1	u_2	u_3
1	x		x	x	
1		x	x		x
1	x			x	x
1		x	x	x	
1	x		x		x
1		x		x	x
Totals	3	3	4	4	4

We are essentially equally distributing one point between t_1 and t_2 , and equally distributing 2 points among u_1 , u_2 , and u_3 . Therefore, with six voters we can achieve a homogeneous outcome using plurality.

The same process can work for larger values of k : We equally distribute one point among t_1, t_2, \dots, t_r and $k-1$ points among u_1, u_2, \dots, u_k using $n = \text{lcm}(r, k)$ voters. Then each candidate t_i will receive $\frac{n}{r}$ points and each u_i will receive $\frac{(k-1)n}{k}$ points. Since $r \geq 2$ and $k \geq 3$, we have $\frac{1}{r} < \frac{k-1}{k}$, and the outcome will be the homogeneous committee $\{u_1, \dots, u_k\}$.

Also note that the same construction will give a homogeneous outcome when using weights other than plurality as long as the total weight W_1 assigned to the t 's is less than the total weight W_2 assigned to the u 's. If $W_2 > W_1$, then the neutrality assumption allows us to switch the assigned weights so that the t 's have total weight W_2 and the u 's have total weight W_1 , and we can then obtain a homogeneous outcome. The one exception occurs if $r < \lceil \frac{k}{2} \rceil$, in which case we cannot apply the neutrality condition. For example, if $k = 5$, $r = 2$, $s = 5$ and the method assigns weights $(5, 5, 1, 1, 1)$ to $(t_1, t_2, u_1, u_2, u_3)$, then we cannot swap the weights assigned to the t 's and u 's since there are only two t 's. In this exceptional case, we will always obtain a diverse outcome although the weights assigned to the two categories are not equal.

We now argue that assigning equal weights to each category will ensure a diverse outcome. First notice that it is impossible to obtain a homogenous committee consisting of candidates from only $\{t_1, \dots, t_r\}$ since $r < k$. The argument that we will obtain a diverse committee is deceptively simple, based only on averaging. Suppose that the total weight assigned to each category is W and that we have n voters. Then the average tally for the candidates $\{t_1, t_2, \dots, t_r\}$ is $\frac{Wn}{r}$, and the maximum average for any set of k candidates from $\{u_1, u_2, \dots, u_s\}$ is $\frac{Wn}{k}$. Since $r < k$, the average tally for the t 's is greater than the average tally for any subset of k candidates from the u 's. Therefore, the individual tallies of any subset of k candidates from the u 's cannot be greater than the tally of every candidate t_1, \dots, t_r . Further, notice that at least one t must have a higher tally than at least one element of any subset of k candidates from the u 's. Thus, the committee must contain at least one candidate from $\{t_1, \dots, t_r\}$.

Applying this method to Example 4 with a total weight of 2 for each category, our tallies become:

EXAMPLE 5.

# Voters	t_1	t_2	u_1	u_2	u_3
1	2		1	1	
1		2	1		1
1	2			1	1
1		2	1	1	
1	2		1		1
1		2		1	1
Totals	6	6	4	4	4

Thus, the winning committee will include t_1 , t_2 , and one of the candidates u_1 , u_2 , or u_3 depending upon the method chosen for breaking ties, guaranteeing a diverse committee independent of the tiebreaker chosen. One may object to the winning committee consisting of two candidates who appeared on three voters' ballots while excluding two candidates who appear on four voters' ballots, but the outcome does meet the prime directive of producing a diverse committee.

3.3. Case III: $r = k$ and $s = k$. Notice that all previous examples in Ratliff and Saari [13] showing that it is possible to obtain a homogeneous committee apply. That is, we have only removed the requirement that the candidates $\{t_1, \dots, t_k\}$ and $\{u_1, \dots, u_k\}$ are slotted against each other in head-to-head elections. Thus, the results of Theorem 1 showing that unequal weights can give a homogenous committee hold for this unslotted situation. However, we need a different argument that assigning equal weights to both categories will guarantee a diverse outcome since there are some preferences valid in our current scenario that are not valid in slotted elections, such as $\{t_1, u_1, u_2, \dots, u_{k-1}\}$.

Fortunately, the same averaging argument used above immediately shows that when assigning equal weights to both categories, it is impossible for all k candidates in one category to have a larger tally than every candidate in the other category.

3.4. Case IV: $r > k$ and $s \geq k$ (or $s > k$ and $r \geq k$). Every homogeneous outcome that is possible in Case III is also possible in this situation since a ballot from Case III can be padded with dummy candidates to make it an election from this case. Unfortunately, we can get a homogeneous outcome even when using the same weights for both categories.

We will consider a specific example to illustrate the argument that also holds in the general case. Let $k = 3$, $r = 4$ and $s = 3$ so that we are electing a committee of size 3 from candidates $\{t_1, t_2, t_3, t_4\}$ and $\{u_1, u_2, u_3\}$. Consider the following election where every voter selects a diverse committee, and we use the total weight of 2 for each category.

EXAMPLE 6.

# Voters	t_1	t_2	t_3	t_4	u_1	u_2	u_3
1	1	1			2		
1		1	1			2	
1			1	1			2
1	1			1	2		
1	1	1				2	
1			1	1			2
Totals	3	3	3	3	4	4	4

As promised, we obtain a homogeneous committee $\{u_1, u_2, u_3\}$. It is not hard to see why this occurs: We have evenly distributed the 2 points from each voter among the candidates in each category. Since we have four candidates in the first category, the points are more widely dispersed among them than among the candidates in the second category. This same argument works in general.

Suppose we are electing a committee of size k from candidates $\{t_1, t_2, \dots, t_r\}$ and $\{u_1, u_2, \dots, u_s\}$ with $r > k$ and $s \geq k$ where we assign the same total weight to each category. Consider a single voter with the preference $\{t_1, t_2, \dots, t_{k-1}, u_1\}$, which will assign weights w_1, w_2, \dots, w_{k-1} to t_1, t_2, \dots, t_{k-1} , respectively, and weight W to u_1 where $w_1 + \dots + w_{k-1} = W$.

We will create an election with $k \cdot (k + 1)$ voters that will give total weight $k \cdot W$ to each candidate t_1, t_2, \dots, t_{k+1} and $(k + 1) \cdot W$ to each candidate u_1, \dots, u_k resulting in the selection of the homogeneous committee $\{u_1, u_2, \dots, u_k\}$. We rotate the weights through all candidates, but we make no assumptions that the weights w_1, w_2, \dots, w_{k-1} are equal. Notice that this construction is valid due to our initial neutrality assumption. We will form k blocks of $k + 1$ voters, and we can safely ignore candidates t_{k+2}, \dots, t_r and u_{k+1}, \dots, u_s .

EXAMPLE 7.

#	t_1	t_2	t_3	t_4	\dots	t_{k-1}	t_k	t_{k+1}	u_1	u_2	u_3	u_4	\dots	u_k
1	w_1	w_2	w_3	w_4	\dots	w_{k-1}			W					
1		w_1	w_2	w_3	\dots	w_{k-2}	w_{k-1}			W				
1			w_1	w_2	\dots	w_{k-3}	w_{k-2}	w_{k-1}			W			
1	w_{k-1}			w_1	\dots	w_{k-4}	w_{k-3}	w_{k-2}				W		
\vdots					\vdots									
1	w_3	w_4	w_5	w_6	\dots		w_1	w_2						W
1	w_2	w_3	w_4	w_5	\dots			w_1	W					
Totals	W	W	W	W	\dots	W	W	W	$2W$	W	W	W	W	W

Notice this block assigns a total of W to every candidate except for u_1 , which receives $2W$. We can mimic this construction for a total of k blocks where the i th block assigns a weight of W to every candidate except for u_i , which receives $2W$. The cumulative effect will be an election that gives a total of $k \cdot W$ to t_1, \dots, t_{k+1} and $(k + 1) \cdot W$ to u_1, \dots, u_k , resulting in the selection of a homogeneous committee $\{u_1, \dots, u_k\}$.

Therefore, we have proved the following:

THEOREM 2. *Suppose that we are electing a k -person committee with candidates from two different categories with r and s members and that there is a universal intent shared by all voters to elect a committee with representatives from each of*

these two categories. To avoid trivial or empty scenarios, assume that $r > 1$, $s > 1$, and $r + s > k$.

To reflect the diversity criterion, assume that an admissible ballot must have at least one candidate from each category. Further, assume that the voting method assigns weights to the candidates and is neutral with respect to the categories and candidates.

- (1) If $r < k$, $s < k$, and $r + s > k$, then any method will meet the diversity criterion.
- (2) If $r < k$ and $s \geq k$ (or $s < k$ and $r \geq k$), then the diversity criterion will always be met if the sum of weights assigned to the candidates of each category are equal. If $r \geq \lceil \frac{k}{2} \rceil$, then the statement becomes “if and only if”.
- (3) If $r = k$ and $s = k$, then the diversity criterion will always be met if and only if the sum of weights assigned to the candidates of each category are equal.
- (4) If $r > k$ and $s \geq k$ (or $s > k$ and $r \geq k$), then it is impossible to guarantee that the diversity criterion will always be met.

In summary, we can guarantee a diverse committee by placing certain restrictions on the number of candidates in each category and using equal weights for both categories. Note that one interesting consequence is that we have a way to structure an election to reflect a universal diversity criterion without explicitly defining the criterion a priori: Create a ballot with $2k$ candidates for the k -person committee. Each voter selects their top-ranked committee and divides their committee into two groups based on their personal diversity criterion. The voting procedure then assigns equal total weights to the two groups for each of the voters. If the voters’ preferences divide the candidates into two disjoint categories, then the method is guaranteed to reflect this in the outcome. In other words, the method detects the universal diversity criterion without it being specified before the election.

3.5. Probabilities. Our motivation for exploring different weighted methods is that plurality can give a homogeneous outcome. Theorem 2 provides a means to avoid these undesirable outcomes, but just as in the case of slotted elections in Section 2.2, we can ask how likely it is that plurality will give this outcome. We will calculate the probabilities assuming IAC for some small values of k using Ehrhart polynomials and compare to the results from Section 2.2 for elections of Case III.

For unslotted elections with $k = r = s = 3$, there are considerably more diverse committees than in the slotted case. Specifically, there are 18 diverse committees in the unslotted case rather than six in the slotted case. The linear inequalities corresponding to the homogeneous outcome of $\{u_1, u_2, u_3\}$ determines a polytope in \mathbb{R}^{17} . As expected, the computations in this case are much more extensive than in the slotted case. Approximately seven hours of computer computation were needed to determine that the Ehrhart polynomial has period 1260 with leading coefficient $\frac{436480777}{9486781262199521280000000}$. Thus, the probability of non-diverse outcome using plurality is

$$\frac{436480777}{9486781262199521280000000} \cdot 17! \cdot 2 \approx 0.03273$$

Notice this is less than the probability of 0.0625 for the slotted case that we calculated in Section 2.2. We will give some intuition why the value is less in the next section.

For $k = r = s = 4$, there are 68 diverse committees in the unslotted case, rather than 14 in slotted elections. The polytope for the Ehrhart polynomial for $\{u_1, u_2, u_3, u_4\}$ will lie in \mathbb{R}^{67} , which creates a problem that is much too computationally intensive to complete in a reasonable amount of time. However, we expect the value to be less than 0.05176 found for the slotted case found in Section 2.2 for reasons explained in the next section. Larger values of k , or examples from Case II or IV, are also too computationally intensive to compute using the current tools.

3.6. Geometric intuition. Although we cannot calculate the probabilities directly, we can develop a geometric framework to provide some intuition about why the probabilities change and suggest what we should expect for larger committees.

Consider an unslotted election to select a committee of size $k = 3$ with candidates coming from the two categories $\{t_1, t_2, t_3\}$ and $\{u_1, u_2, u_3\}$. By placing the candidates in the order $(t_1, t_2, t_3, u_1, u_2, u_3)$, we can associate any ballot with a point in \mathbb{R}^6 . For example, the ballot $\{t_1, u_1, u_2\}$ corresponds to the point $(1, 0, 0, 0, 1, 1)$. There are $\binom{6}{3} = 20$ possible committees, with two homogeneous and the remaining 18 diverse. We will denote the points corresponding to the diverse committees by p_1, p_2, \dots, p_{18} , and the homogeneous committees by p_{19} and p_{20} . When there is no chance of confusion, we will also use p_i to represent the committee corresponding to the point.

If we take a weighted average, then we can view our election tallies as determining a point p in the convex hull of $\{p_1, p_2, \dots, p_{18}\}$. For example, using the election in Example 2 gives

$$3(t_1, u_2, u_3), 3(t_2, u_1, u_3), 5(t_3, u_1, u_2), 1(t_1, t_2, u_2)$$

Using plurality, we have

$$\begin{aligned} p &= \frac{1}{12}((3, 0, 0, 0, 3, 3) + (0, 3, 0, 3, 0, 3) + (0, 0, 5, 5, 5, 0) + (1, 1, 0, 0, 1, 0)) \\ &= \frac{1}{12}(4, 4, 5, 8, 9, 6) \end{aligned}$$

By inspection we can see that the last three components, corresponding to u_1 , u_2 , and u_3 , have the largest value, giving the outcome of $\{u_1, u_2, u_3\}$. However, there is also a geometric interpretation to these results: The point p is closer to $(0, 0, 0, 1, 1, 1)$ than the other 19 possible committees. One way to see this is as follows. Since all of the components of p are non-negative, we can identify the three largest components by maximizing the dot product $p \cdot p_i$ for $1 \leq i \leq 20$. Since $p \cdot p_i = |p| |p_i| \cos(\theta) = |p| \sqrt{3} \cos(\theta)$, we want to maximize $\cos(\theta)$. Therefore, we want to *minimize* θ , or find the p_i such that the angle between p and p_i is smallest.

Let us consider the hyperplane consisting of points equidistant from p_i and p_j . Then any point on the p_i side of the hyperplane, when considered as a vector based at the origin, will have a smaller angle with p_i than with p_j . Thus, if we consider all such hyperplanes between all pairs of points in $\{p_1, \dots, p_{20}\}$, then we have divided the positive orthant of \mathbb{R}^6 into cone-like regions about each p_i . These regions determine the outcomes of the election independent of the weighted method used.

Therefore, plurality can give a homogeneous outcome because the convex hull determined by $\{p_1, p_2, \dots, p_{18}\}$ intersects the regions corresponding to homogeneous outcomes p_{19} and p_{20} . However, by using the same weights for each category, we are changing the points that determine the convex hull. For example, rather than $(1, 1, 0, 0, 0, 1)$ we would use $(1, 1, 0, 0, 0, 2)$. This deformation adjusts the convex hull so that it no longer intersects the homogeneous regions corresponding to p_{19} and p_{20} , guaranteeing that the outcome is a diverse committee.

This interpretation can also give some insight into the decrease in probability in obtaining a homogeneous outcome when using plurality and moving from the slotted to the unslotted elections for $k = 3$ and three candidates of each category. With the slotted election, there are six diverse committees and two homogeneous committees. With the unslotted elections, there are 18 diverse committees and still only two homogeneous committees. This suggests that the 12 additional diverse committees in the unslotted case expand the convex hull to have a larger portion outside of the homogenous regions.

This can give us some insight into what we should expect for larger values of k as well. With $k = 4$ and four candidates in each category, the slotted case will have 14 diverse committees, two homogeneous committees, and the convex hull will lie in \mathbb{R}^8 . Compared to the $k = 3$ case, it is therefore not surprising that the probability of a homogeneous outcome using plurality decreases from 0.0625 to 0.05176. Further, in the unslotted case, there are $\binom{8}{4} = 70$ possible committees. By the same argument as above, we should expect that the probability of a homogeneous outcome using plurality is lower in the unslotted case than in the slotted case.

Although the direct calculations are too computationally intensive to complete at this time, we should anticipate similar results to hold for larger values of k . That is, the probability of obtaining a homogeneous outcome using plurality will be greater for the slotted case than in the unslotted case for any value of k , and the probabilities will decrease as k increases. Specifically, we should expect the probability of a homogeneous outcome using plurality for $k = 6$ with six candidates in each category to be 5% or less for either the slotted or unslotted case. This diminishing probability could play a role in determining whether or not one chooses to implement this approach.

This framework can also be applied if $r \neq s$, but there will be more homogeneous committees that will not be as symmetrically located in the positive orthant as in the $r = s$ case. This makes the comparison for different values of r , s and k somewhat more subtle and less fruitful.

4. Three or more categories

We will now consider elections where there are $N \geq 3$ diversity categories, such as selecting a committee that consists of assistant, associate, and full professors. We will first consider $N = 3$ categories, although similar arguments easily extend to larger values of N . For simplicity to avoid many separate cases, we will assume that there are the same number of candidates in each of the N categories.

4.1. $N = 3$ categories. Let us first consider a very simple example of electing a committee of size $k = 4$ selected from six candidates from $N = 3$ categories, $\{a_1, a_2\}$, $\{b_1, b_2\}$, and $\{c_1, c_2\}$. Not surprisingly, it is possible for the plurality outcome to include candidates from only two of the three categories.

EXAMPLE 8.

# Voters	a_1	a_2	b_1	b_2	c_1	c_2
1	x		x	x	x	
1		x	x	x		x
1	x		x		x	x
1		x		x	x	x
Totals	2	2	3	3	3	3

The problem is the same as the one we have seen before. By assigning unequal weights to the categories, we are able to limit the weights in one category while favoring the candidates in other categories in order to exclude all candidates from the first category.

For $N = 3$ and $k > 4$, the trivial solution is to restrict the number of candidates in each category to fewer than $\lceil \frac{k}{2} \rceil$ candidates so that it is impossible to select k candidates without including at least one candidate from each category. This is analogous to the triangular region in Figure 1 below the line $r + s = k$. However, this is a very unsatisfactory solution because this severely limits the total number of candidates. For example, an election to select a committee of size $k = 7$ would have at most $\lceil \frac{7}{2} \rceil - 1 = 3$ candidates per category, giving nine candidates in total. This effectively changes the election into selecting a few candidates to exclude rather than a choice of which to include. Notice we exclude $k = 3$ or $k = 4$ since $\lceil \frac{k}{2} \rceil - 1 = 1$ and each category would have only a single candidate, giving three candidates total. To see the limitations of this restriction, consider the following table.

EXAMPLE 9.

k	$\lceil \frac{k}{2} \rceil - 1$	Total # Candidates	Total # Excluded
5	2	6	1
6	2	6	0
7	3	9	2
8	3	9	1
10	4	12	2

If each category has exactly $\lceil \frac{k}{2} \rceil$ candidates, then a construction similar to Example 8 for $k = 4$ shows that plurality can exclude the first category from the outcome. We can easily extend this to any method that does not assign equal weights to all three categories. Suppose we have a ballot that assigns a total weights of $W_1 \leq W_2 \leq W_3$ to the three categories, where at least one is strict inequality. We then create an election similar to our example in Case IV in Section 3 where we use $\lceil \frac{k}{2} \rceil$ voters to evenly distribute the total weights of W_1 , W_2 , and W_3 among the candidates in Category 1, Category 2, and Category 3, respectively, and then use $\lceil \frac{k}{2} \rceil$ voters to evenly distribute the total weights of W_1 , W_3 , and W_2 among the candidates in Category 1, Category 2, and Category 3, respectively. This will give us an election of $2 \lceil \frac{k}{2} \rceil$ voters where each candidate in Category 1 receives $2W_1$ points, and each voter in Categories 2 and 3 receives $W_2 + W_3$ points. Since $W_1 \leq W_2$ and $W_1 < W_3$, this ensures that $2W_1 < W_2 + W_3$ and no candidate from Category 1 will be included in the committee.

A nearly identical averaging argument as before also shows that if we use the same total weight for each category and each category has $\lceil \frac{k}{2} \rceil$ candidates, then

it is impossible to exclude one category entirely from the committee. To exclude Category 1, we must include all $\lceil \frac{k}{2} \rceil$ from at least one of the other categories. However, it is impossible for all $\lceil \frac{k}{2} \rceil$ candidates in this category to have tallies larger than each of the $\lceil \frac{k}{2} \rceil$ candidates in Category 1 since both groups have the same average tally.

While providing some way forward, this is still slightly dissatisfying since the elections have fewer than $2k$ candidates so that more candidates are included than excluded. We would merely be adding 3 to the two last columns of the table above. For example, with $k = 7$ we would have four candidates per category and only exclude five of the 12 total.

Similar to Case IV in Section 3, we can also show that if the categories have more than $\lceil \frac{k}{2} \rceil$ candidates each, then using the same total weights in the categories will not guarantee that every category is included in the final outcome. We use $\lceil \frac{k}{2} \rceil \cdot (\lceil \frac{k}{2} \rceil + 1)$ voters to create an election with the same number of points evenly distributed among $\lceil \frac{k}{2} \rceil + 1$ candidates in Category 1 and $\lceil \frac{k}{2} \rceil$ candidates in both Category 2 and 3. This will give $2 \lceil \frac{k}{2} \rceil$ candidates in Category 2 and 3 with a larger tally than every candidate in Category 1, thereby excluding Category 1 from the committee.

Thus, we have proved the following:

THEOREM 3. *Suppose that we are electing a k -person committee, $k \geq 3$, with candidates from three different categories, each with the same number of candidates. Further suppose that there is a universal intent shared by all voters to elect a committee with representatives from each of these categories.*

To reflect the diversity criterion, assume that an admissible ballot must have at least one candidate from each category. Further, assume that the voting method assigns weights to the candidates and is neutral with respect to the categories and candidates.

- (1) *If the number of candidates per category is less than $\lceil \frac{k}{2} \rceil$ and $k > 4$, then any method will meet the diversity criterion.*
- (2) *If there are $\lceil \frac{k}{2} \rceil$ candidates per category, then the diversity criterion will always be met if and only if the sum of weights assigned to the candidates of each category are equal.*
- (3) *If there are more than $\lceil \frac{k}{2} \rceil$ candidates per category, then it is impossible to guarantee that the diversity criterion will always be met.*

These same arguments generalize to $N > 3$ categories with the dividing line being $\lceil \frac{k}{N-1} \rceil$. If there are fewer than $\lceil \frac{k}{N-1} \rceil$ candidates per category, then it is impossible to select k candidates without including at least one candidate from each category. If there are exactly $\lceil \frac{k}{N-1} \rceil$ candidates per category, then any method that does not assign the same total weight to every category can give an outcome that excludes one category using a similar construction as above. Using the same total weights per category in this case must include a candidate from every category by following the same averaging argument: To exclude Category 1 from the output, all $\lceil \frac{k}{N-1} \rceil$ candidates in another category must have a larger tally than every candidate in Category 1, but this is impossible since both categories have the same number of candidates and the same average tally. If there are more than $\lceil \frac{k}{N-1} \rceil$ candidates

per category, then we can mimic the constructions used before to show that any method can give a homogeneous outcome.

THEOREM 4. *Suppose that we are electing a k -person committee, with candidates from $N \geq 3$ different categories where $k \geq N$ and each category has the same number of candidates. Further suppose there is a universal intent shared by all voters to elect a committee with representatives from each of these categories.*

To reflect the diversity criterion, assume that an admissible ballot must have at least one candidate from each category. Further, assume that the voting method assigns weights to the candidates and is neutral with respect to the categories and candidates.

- (1) *If the number of candidates per category is less than $\lceil \frac{k}{N-1} \rceil$ and there are at least k candidates in total, then any method will meet the diversity criterion.*
- (2) *If there are $\lceil \frac{k}{N-1} \rceil$ candidates per category, then the diversity criterion will always be met if and only if the sum of weights assigned to the candidates of each category are equal.*
- (3) *If there are more than $\lceil \frac{k}{N-1} \rceil$ candidates per category, then it is impossible to guarantee that the diversity criterion will always be met.*

While the second part of the theorem does provide methods for guaranteeing that the selected committee will include candidates from every category, in practice this is unlikely to be implemented. The criticism from Section 4.1, while less pronounced than in the $N = 3$ category case, is still evident with $N > 3$ categories. Specifically, more candidates on the ballot are included in the winning committee than are excluded. For example with $N = 4$ categories, selecting a committee of size $k = 9$ would have $\lceil \frac{9}{3} \rceil = 3$ candidates per category, or 12 candidates total, and selecting a committee of size $k = 10$ would have $\lceil \frac{10}{3} \rceil = 4$ candidates per category, or 16 candidates total.

Also note that while it is theoretically feasible to use Ehrhart polynomials to calculate the probability of a homogeneous outcome using plurality, the large number of possible committees places the corresponding polytope in a very high dimensional space, making the required calculations infeasible with current tools.

5. Conclusion

Our primary goal for this paper was to develop methods for electing a committee of size k that respect a universal diversity criterion shared by the electorate. In building off previous work in Ratliff and Saari [13], we restricted to methods that assign points to the individual candidates and only require the voters to indicate their top-ranked committee. When there are two categories determined by the diversity criterion, Theorem 2 provides us with some good news: We create a ballot with k candidates from each category, and we are guaranteed a diverse outcome by assigning the same total weight to the candidates from each category on each voter's ballot. Furthermore, this can be applied even if the diversity criterion is not determined before the election: Create a ballot with $2k$ candidates, and ask each voter to divide their top-ranked committee into two categories representing their personal diversity criterion and assign an equal total weight to each set. If the

preferences of all voters divide the candidates into two disjoint sets, then Theorem 2 guarantees that the outcome will respect this diversity criterion.

However, our results in Theorems 3 and 4 are not nearly so satisfying for three or more categories. The only way to guarantee a diverse outcome within our framework is to limit the number of candidates so severely that it would almost certainly be unacceptable in practice. This suggests that any acceptable method that respects a diversity criterion with three or more categories would require more extensive preferences from the voters and a more elaborate voting method.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, WHEATON COLLEGE, NORTON, MASSACHUSETTS, 02766

E-mail address: `ratliff_thomas@wheatoncollege.edu`

Expanding the Robinson-Goforth system for 2×2 games

Brian Hopkins

ABSTRACT. Robinson and Goforth formalized a system of exchanges to connect all 2×2 ordinal rank games with distinct preferences. We sketch a collection of simplices to expand their system to include all 2×2 ordinal rank games with ties. Also, we revisit some enumeration results and use graph neighborhoods to further analyze their system.

1. Introduction

The most fundamental concept in noncooperative game theory is the 2×2 game, where two decision makers each have two choices and the relative payoffs for each player over the four possible outcomes are given in a bimatrix. Prisoners' Dilemma and Chicken are well-known examples, but there are many others, including games that are asymmetric and have ties among some of the players' preferences.

We restrict our attention to 2×2 games with ordinal ranks (which does not change analysis of dominant strategies or Nash equilibria, for instance). In §2, we further require that each player have distinct preferences over the four possible outcomes. Robinson and Goforth counted 144 such games and formalized a system of connecting them [7] which can be described as using the “neighborly transpositions” (12), (23), (34) from the symmetric group S_4 . This connecting system is simpler and more uniform than related earlier attempts, which are more properly taxonomies than networks. We summarize the Robinson-Goforth system, considering it a graph with edges determined by the exchanges, revisiting enumeration results, and offer a new graph theoretic analysis.

In §3, we allow players to have ties in their ranking of possible outcomes. There are many more such games, and it would be helpful to integrate them into the existing system of games with distinct ranks. Games with a single tie for one player can be identified with edges in the Robinson-Goforth graph, but attempts to incorporate other games into the graph have fallen short [4, 8]. We sketch a collection of simplices to hold all 1,413 ordinal rank 2×2 games and illustrate finding the games adjacent or incident to an example.

Although this article is primarily motivated by mathematical structures, we do mention game theoretic motivations throughout.

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2. The Robinson-Goforth system

We use two ways to represent a 2×2 game, a bimatrix (“normal form” table) and a diagram with the ordered pair of rankings for each possibility designating a point and connecting lines indicating the choices that players face. For example, Figure 1 presents an asymmetric game where the row player chooses between the states labeled (1, 4) and (2, 1) should the column player choose the first column, and between the states labeled (3, 3) and (4, 2) should the column player choose the second column. These are indicated by red line segments in the diagram connecting those pairs of points. (Diagrams are condensed such that the origin corresponds to the point (1, 1).) We consider 4 the highest rank, so that in the diagram, the row player’s payoff is optimized by choosing the rightmost point of a pair connected by a red (darker) line segment. The column players’ choices are connected by green (lighter) segments; the higher point is preferred. The state labeled (4, 2) is a Nash equilibrium (being at the right of a red segment and the top of a green segment).

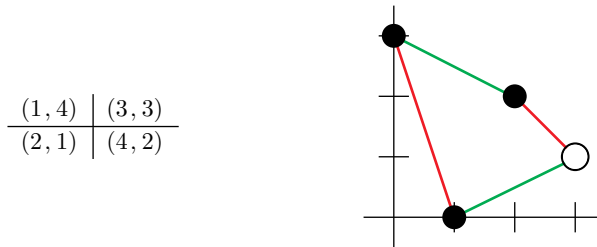


FIGURE 1. A bimatrix and the diagram for an asymmetric game.

2.1. Enumeration. A bimatrix corresponds to an element of the cross product $S_4 \times S_4$; the example in Figure 1 corresponds to the ordered permutation pair $((1, 3, 2, 4), (4, 3, 1, 2))$. In a bimatrix, rows or columns could be switched and still describe the same game (and diagram). This leads Robinson and Goforth to count $24^2/4 = 144$ distinct ordinal rank 2×2 games, an enumeration they confirm by counting directly from ways to build game diagrams [7, §2.4.1].

A more accepted count is 78, which results from allowing the additional symmetry that the row and column players could switch roles and face the same game if the bimatrix is reflected appropriately [6]. To reconcile the counts, of the 78 games, 12 are symmetric (for which the role switching has no effect) and 66 are asymmetric; Robinson and Goforth count the asymmetric pairs separately, giving $12 + 2 \cdot 66 = 144$. We will not here enter into the game theoretic arguments about which count is “correct,” rather we proceed with the 144 count because it provides for more uniform mathematical structures. (By analogy, consider the pedagogy of using a 6 by 6 table to compute probabilities of sums when rolling two six-sided dice despite the fact that, arguably, there are only 21 possibilities—6 pairs and 15 rolls showing unequal numbers.)

Robinson and Goforth enumerate the games by a three digit scheme, abc with integers $1 \leq a \leq 4$, $1 \leq b \leq 6$, and $1 \leq c \leq 6$. They are typically shown in four 6 by 6 tables (by first digit) suggested by the exchanges described below. Indexing is chosen so that Prisoners’ Dilemma has the honor of being game 111.

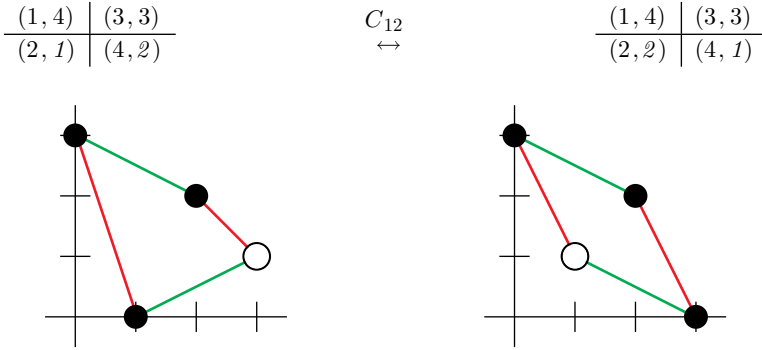


FIGURE 2. An example of exchanging the column player’s lower two preferences (y -coordinates 1 and 2). Notice that the right hand game is Prisoners’ Dilemma.

2.2. Connections. One of the challenges in using game theory to model the interaction of decision makers is determining the ranks of the various possible outcomes. The best and worst possibilities may be clear enough, but it may be unclear how to rank the middle two. With this in mind, Robinson and Goforth connect games by neighborly transpositions that swap adjacent rankings. In particular, there are six exchanges, three per player: R_{12} swaps the 1 and 2 in the row player’s ranking, similarly R_{23} , R_{34} , C_{12} , C_{23} , and C_{34} . For instance, the game from Figure 1 and its image under the C_{12} exchange are shown in Figure 2.

With these exchanges, each game is connected to six others. In other language, let the Robinson-Goforth graph have the 144 games as vertices connected by edges determined by the six transpositions. This gives a 6-regular graph; the neighbors of Prisoners’ Dilemma are shown in Figure 3. (Note that regularity would be lost if asymmetric pairs of game were not considered separately; symmetric games would only have three neighbors.)

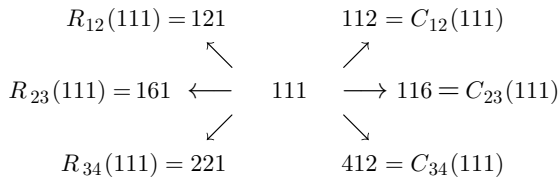


FIGURE 3. The neighbors of Prisoners’ Dilemma, with Robinson-Goforth numbering.

The graph has $(144 \cdot 6)/2 = 432$ edges. Analysis of the adjacency matrix shows that the graph has diameter 6.

As suggested by the game numbering in Figure 3, the four layers of 36 games are generated by application of the exchanges $R_{12}, R_{23}, C_{12}, C_{23}$. Although presented as 6 by 6 tables, the operations “wrap around the edges” such that each layer is a torus. The inclusion of the R_{34} and C_{34} exchanges greatly complicates the picture; Robinson and Goforth determine that the graph has genus 37 [7, §§6.3, 7.2]. Other structures of interest arise from the application of the exchanges $R_{12}, R_{34}, C_{12}, C_{34}$

which include 8 or 16 games and are called, respectively, hotspots and pipes. A hotspot connects sets of four games from two layers, while a pipe connects sets of four games from all four layers.

Earlier structures on 2×2 distinct ordinal rank games were based on game theoretic characteristics. Rapoport, Guyer, and Gordon developed a system of phylum, class, order and genus to organize their 78 games [6]. The genera vary in size from 1 to 13 games, and the system is reminiscent of the phenotypic taxonomy in biology predating the discovery of DNA.

For the most part, practitioners have proceeded as if “each game is an island, studied independently.” Robinson and Goforth claim instead, “There is no island of California and there is no island of Prisoner’s Dilemma.” [2, p.603]. They spend considerable time on the consistency of various game theoretic characteristics with their more uniform system [7]. The notion of distance between games is helpful in practice since many times close games share certain characteristics.

2.3. Neighborhood analysis. The k th neighborhood of a vertex v in a graph consists of all vertices whose distance from v is k . While the Robinson-Goforth graph is regular, i.e., every vertex has the same size first neighborhood, the sizes of higher neighborhoods vary. However, each vertex falls into one of five types determined by sequence of neighborhood sizes, described in Table 1.

type	number	0	1	2	3	4	5	6	average
A	24	1	6	16	32	43	34	12	3.81
B	24	1	6	16	32	47	34	8	3.75
C	24	1	6	19	36	43	30	9	3.67
D	48	1	6	19	39	46	27	6	3.58
E	24	1	6	19	42	46	24	6	3.54

TABLE 1. Five types of vertices determined by neighborhood sizes.

For instance, each of the 24 type A games has 16 games distance 2 from it, 32 games distance 3 away, 43 games 4 away, 34 games 5 away, and 12 games 6 away. Weighting the number of games by their distance gives an average distance of $548/144$ to other games. Each 7-tuple of neighborhood sizes sums to 144. We can elaborate on the graph diameter statistic to note that each game is distance 6 from 6, 8, 9, or 12 other games.

There is additional structure within these five types. For instance, each type A game has four type B and 2 type C games in its first neighborhood, two type A games and 12 type D games in its second neighborhood, etc. This sort of consistency appears for all types at every distance.

These five types of games fit well with the Robinson-Goforth structures hotspots and pipes. In particular, each hotspot consists of four type A and four type B games. The higher average distance for these games is consistent with the fact that each hotspot connects just two of the four layers. Each pipe consists of 4 type C, 8 type D, and 4 type E games. The lower average distance is consistent with each pipe connects games in all four layers. The consistency of these graph theoretic game types with hotspots and pipes suggests that these two structures (as opposed to layers) merit additional attention.

3. Games with ties

There are many situations where players do not have distinct preferences over all possibilities. In Rousseau's classic Stag Hunt, for instance, there are three outcomes: catch the stag, catch a hare (for two of the four possibilities), and catch nothing. In terms of modeling situations, another approach to uncertainty about ranking two outcomes could be to consider the player indifferent between those two options.

There are two related ways to put games with ties into the Robinson-Goforth system. The exchange of 1 and 2, for instance, could be considered a continuous transformation to $(1 + \alpha, 2 - \alpha)$ for $0 \leq \alpha \leq 1$. In terms of ordinal preference,

$$(1 + \alpha, 2 - \alpha) \sim \begin{cases} (1, 2) & \text{if } \alpha < \frac{1}{2}, \\ (1, 1) & \text{if } \alpha = \frac{1}{2}, \\ (2, 1) & \text{if } \alpha > \frac{1}{2}, \end{cases}$$

where \sim denotes ordinal equivalence. However, it is helpful to use (1.5, 1.5) for the case $\alpha = \frac{1}{2}$, the tie halfway along the exchange. The discrete alternative is to consider just these half-exchanges, restricting to $\alpha = \frac{1}{2}$, which we will denote \overline{R}_{12} , etc.

3.1. Symmetric games in three dimensions. As a preliminary step, we review recent work of Robinson and Goforth to model all symmetric ordinal rank 2×2 games in a single structure [2]. (Much of the work described here was done independently at roughly the same time by Sarah Heilig and the author [3].)

To maintain symmetry, the exchanges are combined into three: $S_{12} = R_{12} \circ C_{12}$, etc., with the corresponding half-exchanges \overline{S}_{12} , etc. There are 12 symmetric games with distinct ordinal ranks, and each has three neighbors under these operations. There 18 with one tie for both players (so that each player has three ordinal ranks), 7 with two ties (each player has two ordinal ranks), and 1 with three ties (the fully degenerate game where neither player has any preference over any outcome).

These "inbetween" games with ties can help explain adjacent games in the Robinson-Goforth system with different characteristics. For instance, Prisoner's Dilemma, with its famous Pareto sub-optimal Nash equilibrium, is adjacent via the S_{12} operation to Chicken, which has two Nash equilibria, both with optimal rank sum. Examining the \overline{S}_{12} game between them helps explain this drastic change. This idea dates back to the 1970's, although it was not developed then in full generality. Without explaining Rapoport, Guyer, and Gordon's notation, one can see the same idea being considered.

Equalizing certain payoffs allows us to construct "borderline games." Consider Prisoner's Dilemma (game #12) in which the payoffs associated with $S_1 S_2$ keep decreasing. As long as these payoffs are larger than the payoffs associated with unilateral departure from the natural outcome, the game is still Prisoner's Dilemma. But as soon as the decreasing payoffs become smaller than those associated with unilateral cooperation, the game turns into game #66 (Chicken). So, when the equality is reached, we obtain a "borderline" game on the juncture of Prisoner's Dilemma and Chicken.

Other such borderline games can be obtained similarly. The principal use of equalizing payoffs is to eliminate a particular pressure so as to study the effects of another pressure. [6, p.31]

Figure 4 shows Prisoner’s Dilemma, Chicken, and their “borderline game.” The point (1, 4) transitions through (1.5, 4) before reaching (2, 4). Notice the the

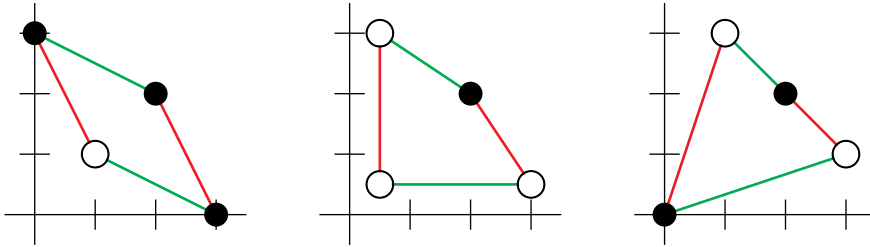


FIGURE 4. Diagrams for Prisoners’ Dilemma on the left, Chicken on the right, and their $\overline{S_{12}}$ game in between. These are the $\alpha = 0, \frac{1}{2},$ and 1 values of the $(1 + \alpha, 2 - \alpha)$ transition.

y -coordinate 4 does not change, as it is not effected by S_{12} . Similarly, the point (4, 1) transitions through (4, 1.5) before reaching (4, 2). The point (2, 2) transitions through (1.5, 1.5) before reaching (1, 1), while the point (3, 3) is unchanged by S_{12} .

This $\overline{S_{12}}$ game helps one understand the change between the very different Nash equilibria characteristics of the two well-known games. The point $(2 - \alpha, 2 - \alpha)$ is a Nash equilibrium for $\alpha \leq \frac{1}{2}$, a range that includes Prisoners’ Dilemma and the borderline game. The points $(1 + \alpha, 4)$ and $(4, 1 + \alpha)$ are Nash equilibria for $\alpha \geq \frac{1}{2}$, a range that includes Chicken and the inbetween game. In this $\overline{S_{12}}$ game, there are three Nash equilibria. At this value $\alpha = \frac{1}{2}$, both players are indifferent between certain choices, corresponding to horizontal and vertical line segments in the diagram. Thinking of α moving from 0 to 1, the inbetween game includes the Pareto non-optimal Nash equilibrium “on its way out” and the two Nash equilibria from Chicken “just starting.”

To include all symmetric games in a single structure, Robinson and Goforth place these games on a polyhedron, a “winged octagon” with two flaps attached to antipodal edges; see Figure 5. Each of 12 triangular faces corresponds to a game with distinct rankings, and adjacent games share an edge. Each of the $(12 \cdot 3)/2 = 18$ edges corresponds to a game where each player has one tie. Thus each distinct rank game is adjacent to three games with a single tie for each player. The games with two ties for each player correspond to the vertices; for instance, $\overline{S_{12}} \circ \overline{S_{23}}$ leads to rankings (1, 1, 1, 2) for both players (which might be considered (2, 2, 2, 4) instead to preserve the rank sum) and this corresponds to the vertex where the lines corresponding to $\overline{S_{12}}$ and $\overline{S_{23}}$ intersect. Each distinct rank game is adjacent to three games with two ties for each player, matching the number of ways to choose two out of three half-exchanges. Each vertex is incident to four or six faces (the six face cases occur where the flaps attach to the octagon). While the geometric figure has eight vertices, the two wingtips are identified, making that single vertex adjacent to both pairs of games on the flaps. See [2] for more details.

It is reasonable to have a three dimensional model for symmetric games. To build a symmetric game bimatrix, there are three degrees of freedom: choose one

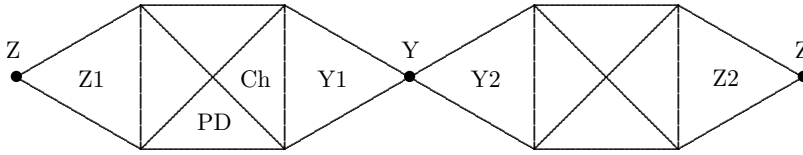


FIGURE 5. Robinson-Goforth structure of symmetric 2×2 games with Prisoners’ Dilemma and Chicken highlighted. Folding the figure so that the points labeled Z coincide gives an octagon with “wings,” where games $Y1$ and $Y2$ are on each side of one wing, $Z1$ and $Z2$ on each side of the other. Further, vertices Y and Z are identified as one vertex since they correspond to the same game with two ties for each player.

rank for a one matching possibility (say the 2’s in the normal form of Prisoners’ Dilemma, shown on the upper right of Figure 2), choose one remaining rank for the other matching possibility (the 3’s in PD), and then the order of the two other ranks in a remaining possibility (the (1, 4) in the upper right) — the game is completely determined at this point since the remaining possibility is forced.

3.2. All games in five dimensions. To consider asymmetric games with ties, we return to the six operations which exchange two subsequent ranks for a single player. Figure 5 shows the nine games arising from R_{12} (applied across rows, where x -values change) and C_{12} (columns, y -values) on Prisoners’ Dilemma and Chicken. To describe Figure 5 more succinctly, we abbreviate Prisoners’ Dilemma by PD and Chicken by Ch. The four corner games have distinct rankings for both players and their Robinson-Goforth numbering is given. The horizontal bar includes the $\overline{C_{12}}$ games where the column player is indifferent between the two lowest options. The vertical bar includes the analogous games for $\overline{R_{12}}$. The game in the middle has two ties, one per player, and can be described as $(\overline{R_{12}} \circ \overline{C_{12}})(PD)$ or $(\overline{R_{12}} \circ \overline{C_{12}})(Ch)$. Figure 4 consists of the lower left, middle, and upper right games of Figure 5.

The two operations used in Figure 5 commute, i.e., $(R_{12} \circ C_{12})(PD) = Ch = (C_{12} \circ R_{12})(PD)$. Of the $\binom{6}{2} = 15$ possible pairs of operators, eleven commute: nine of the form R_{wx}, C_{yz} , and also R_{12}, R_{34} and C_{12}, C_{34} . Each such pair will generate nine games as in Figure 5; compare this to the degree 4 vertices in the “winged octahedron” described in §3.1. In the remaining four pairs, the two operations satisfy a braid relation, e.g., $R_{12} \circ R_{23} \circ R_{12} = R_{23} \circ R_{12} \circ R_{23}$. Compare the degree 6 vertices in §3.1.

To include all games in a single structure, one might begin by embedding the Robinson-Goforth graph on a surface and create a map by expanding the vertices to regions; see [4, 8] for work in this direction. Edges between vertices become border edges between regions, and there are now vertices available to extend the model. This is suggested by Figure 5, where the gray edges are associated with games where one player has one tie in ranks, and the intersection of the gray edges is associated to a game with two ties. Since each distinct rank game has six neighbors, the regions are hexagons. The graph has 432 edges, which exactly accommodates the games with a single tie (all enumeration results for games with ties match [1] when adjusted to not allow the player reversal symmetry). However, there are not

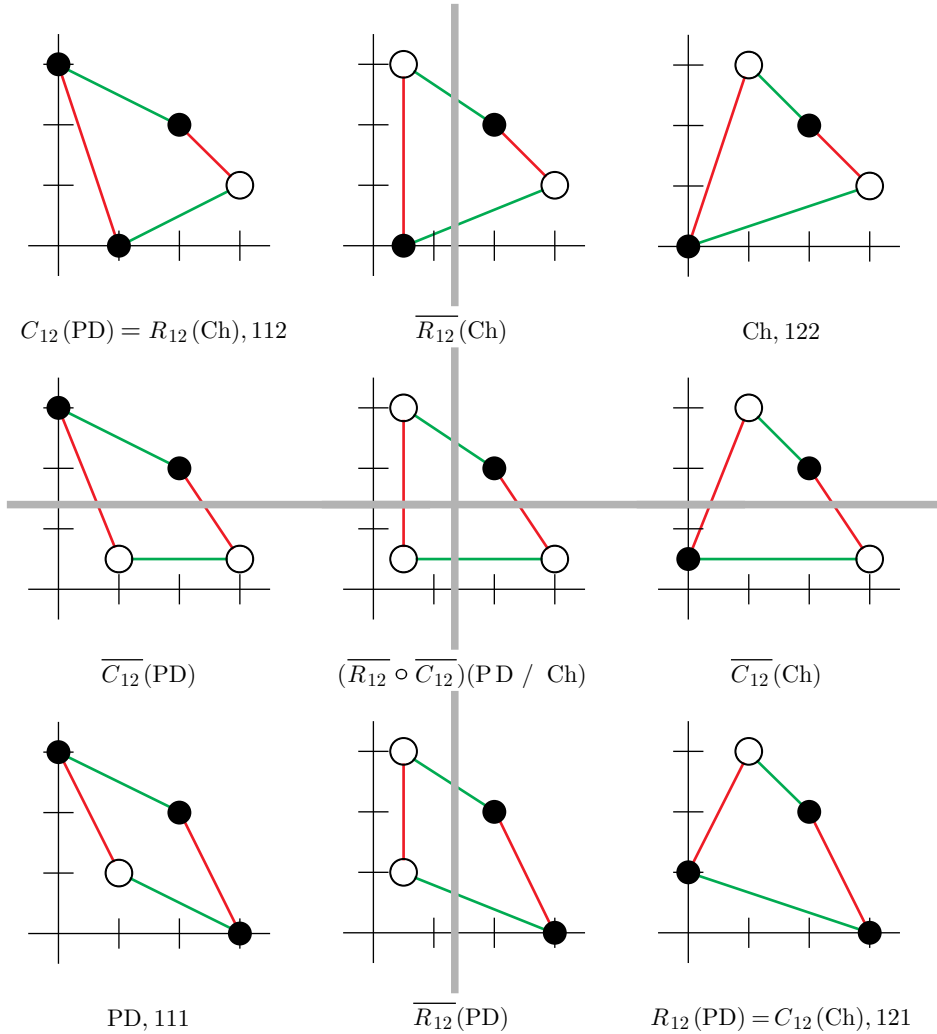


FIGURE 6. Games in between Prisoners' Dilemma and Chicken.

enough vertices for games with two ties: A distinct rank game should be adjacent to 15 games with two ties corresponding to possible pairs of the six exchanges, but a hexagon is bounded by six vertices. Moreover, there are no additional structures in the map to handle games with more than two ties.

Upon reflection, a three dimensional model does not have enough degrees of freedom for this goal. To build an asymmetric game bimatrix, place the row player's lowest rank possibility in the upper left, then there are two choices for where to place other ranks before the last is forced. There are three choices to be made for placing the column player's ranks before the last is forced. This suggests a total of five degrees of freedom.

A mathematical candidate that accommodates all ordinal rank 2×2 games with all appropriate adjacencies is a collection of 5-dimensional simplices. Each

distinct rank game corresponds to a 5-simplex, a five dimensional analogue of a triangle and tetrahedron. Each 5-simplex is incident to

- six 4-simplex boundaries corresponding to games with one tie,
- $\binom{6}{2} = 15$ tetrahedra corresponding to games with two ties,
- $\binom{6}{3} = 20$ triangles corresponding to games with three ties,
- $\binom{6}{4} = 15$ edges corresponding to games with four ties, and
- $\binom{6}{5} = 6$ vertices corresponding to games with five ties,

leaving again the fully degenerate six tie game where neither player has any preference over any outcome. How to piece together the 144 simplices is not clear, though; from the variation in Table 1, care has to be taken to give the desired neighborhood sequences. The collection does not constitute a simplicial complex, as the desired vector $f = (1, 10, 70, 264, 492, 432, 144)$ (where f_i for $0 \leq i \leq 6$ counts the number of games with $6 - i$ ties, counts corroborated by [1]) fails to satisfy Kruskal-Katona theorem to be the f -vector of a simplicial complex. Perhaps identification of components, as with vertices in §3.1, can maintain some geometric manifestation.

To suggest the value of this 5-dimensional program, we determine the incidences of a game considered in [5], where Kilgour and Fraser were providing examples of placing games with ties in a proposed taxonomy. The game is described by the bimatrices in Table 2. The two bimatrices present two representations of the same ties, as per the discussion at the beginning of §3. The bimatrix on the left gives ordinal preferences in positive integers starting form 1. The bimatrix on the right gives adjusted values (using half-integers) so that the rank sum of 10 for each player remains constant.

$$\begin{array}{c|c} (1, 3) & (1, 2) \\ \hline (1, 1) & (2, 1) \end{array} \qquad \begin{array}{c|c} (2, 4) & (2, 3) \\ \hline (2, 1.5) & (4, 1.5) \end{array}$$

TABLE 2. The game G with three ties: two for the row player and one for the column player.

This game G with three ties corresponds to a triangle in the geometric model; to which 5-simplices is it incident? Consider starting from a distinct rank game to which ties are made to arrive at G . Since the row player ranks in G are $(1, 1, 1, 2)$ (or $(2, 2, 2, 4)$), the operations \overline{R}_{12} and \overline{R}_{23} must be applied in some order to the distinct rank game, leaving a single higher rank for the row player in G . Similarly, since the column player ranks are $(1, 1, 2, 3)$, the operation \overline{C}_{12} must be applied.

To find a distinct rank game that reduces to G , we “untie” the ties in any order, such as given in Table 3. (Notice that both ordinal sum representations, on the top row, and constant rank sum representations, on the bottom row, have a similar disadvantage: The change in values from breaking a tie can change the values of other options. E.g., in the top row, the reverse image of \overline{R}_{23} takes the row player values from $(1, 1, 1, 2)$ to $(1, 1, 2, 3)$, changing the value of the most preferred object from 2 to 3, even though it was not part of the tie. Similarly, in the bottom row, the values for the row player change from $(2, 2, 2, 4)$ to $(1.5, 1.5, 3, 4)$.)

Thus the game at the right hand side of Table 3, game 121 in the Robinson-Goforth numbering and the lower right game of Figure 5, reduces to G under

$\frac{(1, 3)}{(1, 1)} \mid \frac{(1, 2)}{(2, 1)}$	$\frac{(1, 3)}{(1, 1)} \mid \frac{(2, 2)}{(3, 1)}$	$\frac{(1, 4)}{(1, 2)} \mid \frac{(2, 3)}{(3, 1)}$	$\frac{(2, 4)}{(1, 2)} \mid \frac{(3, 3)}{(4, 1)}$
$\frac{(2, 4)}{(2, 1.5)} \mid \frac{(2, 3)}{(4, 1.5)}$	$\frac{(1.5, 4)}{(1.5, 1.5)} \mid \frac{(3, 3)}{(4, 1.5)}$	$\frac{(1.5, 4)}{(1.5, 2)} \mid \frac{(3, 3)}{(4, 1)}$	$\frac{(2, 4)}{(1, 2)} \mid \frac{(3, 3)}{(4, 1)}$

TABLE 3. From G , possible reverse images for $\overline{R_{23}}, \overline{C_{12}}, \overline{R_{12}}$, in sequence, resulting in a distinct rank game. The top rows uses ordinal rank, the bottom row uses values adjusted to have constant rank sum.

$\overline{R_{12}}, \overline{R_{23}}$, and $\overline{C_{12}}$. To determine the other distinct rank games that reduce to G under these three half-exchanges, apply the corresponding maps R_{12}, R_{23}, C_{12} in all possible combinations to game 121. This orbit gives the twelve games 1b1 and 1b2 for $1 \leq b \leq 6$, two columns of Robinson and Goforth’s layer 1. One can interpret this as follows: Exactly these twelve games among the distinct rank games share certain characteristics with the game G .

Similar reasoning determines the number of games, partitioned by ties, that either reduce to our reduce from G , shown in Table 4. Geometrically, this means the triangle corresponding to G is incident to 12 5-simplices, 18 4-simplices, 8 tetrahedra, and, as one would hope, contains 3 edges and 3 vertices.

distinct rank games	12
one tie games	18
two tie games	8
four tie games	3
five tie games	3
six tie game	1

TABLE 4. Number of games, by number of ties, that reduce to or reduce from the three tie game G of Table 2.

Although the collection of 5 and lower dimensional simplices to model all 2×2 ordinal rank games has only been sketched here, we hope this example with G provides a compelling proof of concept.

We close with additional game-theoretic motivation for considering half-exchanges and further refinements. In a collection of essays on political economics, Streek and Thelen argue that the abrupt “punctuated equilibrium models” are insufficient to model current developments.

The biases inherent in existing conceptual frameworks are particularly limiting in a time, like ours, when incremental processes of change appear to cause gradual institutional transformations that add up to major historical discontinuities. As various authors have suggested, far-reaching change can be accomplished through the accumulation of small, often seemingly insignificant adjustments (e.g., ‘tipping points’). [9, p. 8]

An initial step to providing game theoretic structure for these gradual transformations is the refinement of exchanges to half-exchanges or continuous exchanges. The full array of 2×2 ordinal rank games, symmetric and asymmetric, with and without ties, are then connected. The simplicial model proposed here could provide a structure for better understanding these connections.

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SAINT PETER'S UNIVERSITY, JERSEY CITY, NEW JERSEY
E-mail address: bhopkins@saintpeters.edu

Cooperation in n -player repeated games

Daniel T. Jessie and Donald G. Saari

ABSTRACT. After reviewing the basics of a new way (Jessie and Saari (2013)) to decompose single-shot games into their strategic and behavioral parts, it is shown how this decomposition provides new results and insights into the nature of repeated interactions. In particular, for any two-strategy $n \geq 2$ player game, it is shown how to obtain a complete characterization of when a specified outcome is sustainable in an infinitely repeated setting with respect to standard choices of punishment. This characterization is given by a simple relationship between strategic and behavioral components. Furthermore, by showing how this analysis extends issues raised by simplified 2-player scenarios to a $n \geq 3$ player setting, it follows that results obtained for a 2-player analysis need not hold for the n -player case even under generous assumptions.

1. Introduction

When modeling events from the social sciences, it could be that no player wants to suffer the consequences of a specified Nash equilibrium. An illustration would be competing countries where the strategic structure leads to the dominant strategy outcome of a war. Fortunately, and as it is well-known, there are equilibria in a repeated setting that differ from the single-shot Nash equilibria. This fact underscores an advantage of repeated games (i.e., games involving repeated interactions) should settings exist where an appropriate strategy can lead to cooperation.

A natural illustration is the following Prisoner's Dilemma game \mathcal{G}_1 ,

$$(1) \quad \mathcal{G}_1 = \begin{array}{|cc|cc|} \hline 4 & 4 & -2 & 6 \\ \hline 6 & -2 & 0 & 0 \\ \hline \end{array}.$$

Although the single-shot Nash strategy is the limited Bottom-Right (BR), it is possible to enjoy the Pareto superior outcome in the Top-Left (TL) corner if TL becomes an equilibrium when the game is repeated infinitely often. But TL is *not* a Nash equilibrium of a single-shot play, so a theoretical concern is to identify what aspects of the game permit cooperation. A theme initiated here is to use a recently developed decomposition of games (Jessie and Saari [3]) to identify the precise source of a game's cooperative structure, which arises only with repeated

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interactions. An interesting consequence is how a game's Nash structure assumes different roles in the analysis with changing opportunities for cooperation.

A second theme involves where a two-person model is used to examine particular settings even should a more realistic modeling involve games with three or more players. As an example while this paper is being written, a contemporary international difficulty involves the countries of Israel and Iran where a natural objective is to determine whether a repeated game could create a cooperative solution rather than an uncomfortable Nash outcome of, say, war. But for the game to capture any sense of reality, another major player, the United States, must be included.

Including the US introduces a technical problem: Two-player repeated games already can be complicated to analyze, so three-player games can be daunting. A natural approach to circumvent this added complexity is to speculate whether the essence of the more realistic three-player game can be captured with a two-player one. With the Iran-Israel example, for instance, one might argue that two of the players (the US and Israel) have fairly common objectives, which makes it reasonable to suspect that a two-player analysis suffices.

The theoretical issue is to determine whether this reasonably common approach is correct: To do so, we examine relationships between two and three player games. As we prove, a two-player game *need not* capture the realities of a three-player game. As an illustration, Brams and Kilgour [1] recently showed that by giving Israel the ability to detect Iran's strategy before choosing its own strategy, a cooperative outcome can be reached in several 2×2 games. We show, however, that once the United States is added as a third player, results based on 2×2 games need not extend to the more realistic three-player settings. Instead, a surprising conclusion is that the ability to detect Iran's strategy might lead to *less* cooperation.

A third theme is motivated by common and continuing international events; this is where countries can use their resources to change the game's structure. A game's strategic structure can be significantly altered, for instance, should a country develop a new bomb or delivery missiles. (The need to avoid these kinds of serious shifts in strategic structures explains the efforts to have nuclear non-proliferation treaties.) Our interest centers on other kinds of modifications, such as the possibility of an adversary's admission into a trade organization or scientific/cultural exchanges. These rearrangements need not affect the game's strategic aspects, but they could make cooperation more attractive in a repeated setting. To examine this possibility, we analyze how to combine the strategic behavior in a repeated game with modifications in the behavioral aspect of a game's structure (i.e., the level of inducement) to attain and sustain cooperation. Answers are based on our decomposition of games.

A way to describe further possible applications is to indicate how our approach differs from other methods. A popular, current approach was developed by Robinson & Goforth [5], and extended by Hopkins [2], to classify the different types of 2×2 interactions. This approach, along with that of Brams and Kilgour [1], should be viewed as classifying types of games rather than as providing new tools to analyze them. As such, these results tend to be local in nature, computationally intensive, do not naturally extend beyond the 2×2 case, and can discard relevant information about payoffs. While our approach also provides a classification of games, the intent is to provide new tools to analyze games.

There are, of course, many ways to decompose games. (For a description along with references, see Jones [4].) While discussing these interesting approaches would detract from the purpose of our paper, some comments are appropriate. For instance, a commonly used choice is to decompose a game into its cooperative and zero-sum components such as

$$(2) \quad \begin{array}{|c|c|c|c|} \hline 6 & 6 & 0 & 4 \\ \hline 4 & -4 & 2 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 6 & 6 & 2 & 2 \\ \hline 0 & 0 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 0 & 0 & -2 & 2 \\ \hline 4 & -4 & 1 & -1 \\ \hline \end{array}$$

This zero-sum decomposition is valuable for several purposes (e.g., bargaining solutions), but it (as with other decompositions) can radically alter the game's strategic structure. This reality is illustrated with Eq. 2, where the game on the left has as mixed and two pure (TL, BR) Nash strategies, but the zero sum component (second bimatrix after the equal sign) has the dominant BR strategy. An important feature of our decomposition is that it retains the original game's strategic structure.

An advantage our decomposition has over a point-wise analysis is that it emphasizes the structure of the space of games (denoted by \mathbb{G}). In this way, it becomes possible to identify *classes of games* with desired properties rather than relying on specific choices (a common approach) to carry out an analysis. In this way we can determine whether obtained conclusions reflect properties of only specifically selected games, or whether they identify a more general behavior. What underscores the importance of this concern is that this global approach also highlights the potential sensitivity of results to small perturbations of a game. For instance, we show below with an example that conclusions derived from a particular choice of payoffs need not hold for other games even if their payoff entries are "close".

Other advantages of the decomposition include demonstrating how non-strategic information can affect conclusions in a repeated setting and examining differences between the repeated and single-shot equilibria. This relationship between strategic and non-strategic information underscores the role of the discount factor δ ; as demonstrated, the δ discount factor interacts with changes in what we call the game's behavioral component. Finally, as this decomposition permits comparing strategies, it can be used to identify all games for which a particular strategy will "out-perform" another strategy.

1.1. Decomposition. While details, proofs, and the motivation for our decomposition are in [3], a brief introduction is given here with the Eq. 1 game \mathcal{G}_1 . The goal is to divide the game into the component \mathcal{G}_1^N that contains all information needed to determine all possible Nash strategic behavior, component \mathcal{G}_1^B that, from a behavioral perspective, contains no Nash information but can change the dynamics of the game in other ways, and a kernel term \mathcal{G}_1^K that merely modifies entries. The important fact is that this decomposition is unique and can be used with all games.

To find \mathcal{G}_1^N from a given game \mathcal{G} , consider the two matrix entries left to a player after specifying a pure strategy for each of the other players. With \mathcal{G}_1 , for example, if L is specified for Player 2, then the two entries for Player 1 are "4" by playing T and "6" by playing B. Replace each entry by how it differs from their average of $(4 + 6)/2 = 5$. That is, replace the 4 with $4 - 5 = -1$ and 6 with $6 - 5 = 1$. Doing this for all players and options leads to

$$\mathcal{G}_1^N = \begin{array}{|c|c|c|c|} \hline -1 & -1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ \hline \end{array}$$

What remains is a listing of the computed averages, or

$$\mathcal{G}_1^A = \mathcal{G}_1 - \mathcal{G}_1^N = \begin{array}{|c|c|c|c|} \hline 5 & 5 & -1 & 5 \\ \hline 5 & -1 & -1 & -1 \\ \hline \end{array}.$$

The kernel component \mathcal{G}_1^K replaces each \mathcal{G}_1 entry for a player by the average of all of the player’s \mathcal{G}_1 entries. For \mathcal{G}_1 , each player’s average is $(4 - 2 + 0 + 6)/4 = 2$. The behavioral component is $\mathcal{G}_1^B = \mathcal{G}_1^A - \mathcal{G}_1^K$. Thus, $\mathcal{G}_1 = \mathcal{G}_1^N + \mathcal{G}_1^B + \mathcal{G}_1^K$ where

$$(3) \quad \mathcal{G}_1^B = \begin{array}{|c|c|c|c|} \hline 3 & 3 & -3 & 3 \\ \hline 3 & -3 & -3 & -3 \\ \hline \end{array}, \quad \mathcal{G}_1^K = \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 \\ \hline \end{array}$$

Notice; \mathcal{G}_1^B and \mathcal{G}_1^K contain no Nash information of any kind. This is because the rows in each matrix are the same for the row player, the columns are identical for the column player, so *individual* actions taken by a player cannot affect the player’s \mathcal{G}^B payoff; selection of \mathcal{G}^B payoffs requires cooperative behavior. In turn, this \mathcal{G}^A property means that \mathcal{G}_1^N contains all of \mathcal{G}_1 ’s Nash information and \mathcal{G}_1^B contains all other aspects about \mathcal{G}_1 that affect behavior.

A theme developed below reflects the need to analyze disagreements between the strategic component \mathcal{G}^N (where the Nash outcomes are based on individual actions) and the behavioral \mathcal{G}^B component (where outcomes require cooperative behavior). The \mathcal{G}_1 Prisoner’s Dilemma captures this conflict; the Pareto superior entry of the behavioral \mathcal{G}_1^B , which is TL, coincides with the location of the Pareto inferior term for the strategic \mathcal{G}_1^N . This disagreement imposes a stress between the game’s cooperative and strategic interests.

To appreciate the effect of the mismatch, recall that *individual strategic actions* affect only \mathcal{G}_1^N outcomes; they cannot influence or attain the \mathcal{G}_1^B desired outcome. Thus a cooperative action (i.e., to ensure TL) must counter individual strategic actions to extract the larger \mathcal{G}_1^B outcome (which generates the larger \mathcal{G}_1 outcome). More generally and for any game \mathcal{G} , all collective and cooperative actions admitted by \mathcal{G} are strictly based on the \mathcal{G}^B structure and how it compares with \mathcal{G}^N . As such, it is the \mathcal{G}^B component that plays a central role in addressing our described themes of cooperation beyond strategic behavior.

The decomposition for a three player game, where each has two strategies, is the same. To illustrate, consider

$$(4) \quad \mathcal{G} : \quad \text{Front} = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 3 & 5 & 4 & 1 & 2 \\ \hline 2 & 2 & 1 & 0 & 4 & 3 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & 2 & 1 & 4 & -2 & 2 \\ \hline 1 & 4 & 5 & 8 & 6 & 7 \\ \hline \end{array}$$

The “Front” and “Back” labels correspond to Player 3’s strategy choices. Although not labeled, “Top/Bottom” and “Left/Right” refer to Player 1’s and Player 2’s strategies, respectively. So the payoffs for the three players should they play Bottom-Left-Back, or BLBa, are 1, 4, 5, respectively.

To find \mathcal{G}^N , select a strategy for two of the players, and then replace the third player’s identified entries by how they differ from their average. If T for Player 1 and L for Player 2 are specified, for instance, then Player 3’s two entries are 5 from “Front” and 1 from “Back.” Replace each term with how it differs from the average of 3. Doing so for all strategies and players, it follows that \mathcal{G}^N is

$$\text{Front} = \begin{array}{|c|c|c|c|c|c|} \hline -1 & 1 & 2 & 2 & -1 & 0 \\ \hline 1 & -1 & -2 & -2 & 1 & -2 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & -2 & -2 & -2 & 0 \\ \hline -1 & -1 & 2 & 2 & 1 & 2 \\ \hline \end{array}$$

As true in general, the \mathcal{G}^N component contains *all information* needed to determine everything about \mathcal{G} 's Nash strategic structure; nothing else is needed. To see that \mathcal{G}^N has extracted all of the strategic information, notice that there is no difference in T or B for Player 1, L or R for Player 2, or F or Ba for Player 3 in

$$\mathcal{G}^A = \mathcal{G} - \mathcal{G}^N : \text{Front} = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 2 & 3 & 5 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|} \hline 2 & 0 & 3 \\ \hline 2 & 5 & 3 \\ \hline 6 & 0 & 2 \\ \hline 6 & 5 & 5 \\ \hline \end{array}.$$

This means that \mathcal{G}^N has extracted all of the \mathcal{G} strategic information.

The average entry for players 1, 2, and 3, is, respectively, $\kappa_1 = \frac{11}{4}$, $\kappa_2 = \frac{5}{2}$, $\kappa_3 = \frac{13}{4}$; each matrix entry in \mathcal{G}^K has these three terms. The \mathcal{G}^B matrix is obtained by replacing the j^{th} player's \mathcal{G}^A entry by how it differs from κ_j . So the Front matrix for \mathcal{G}^B is

$$\text{Front} = \begin{array}{|c|c|c|} \hline 1 - \frac{11}{4} & 2 - \frac{5}{2} & 3 - \frac{13}{4} \\ \hline 1 - \frac{11}{4} & 3 - \frac{5}{2} & 3 - \frac{13}{4} \\ \hline 2 - \frac{11}{4} & 2 - \frac{5}{2} & 2 - \frac{13}{4} \\ \hline 2 - \frac{11}{4} & 3 - \frac{5}{2} & 5 - \frac{13}{4} \\ \hline \end{array}$$

In general and by construction,

$$(5) \quad \mathcal{G} = \mathcal{G}^N + \mathcal{G}^B + \mathcal{G}^K.$$

The general form of each of these matrices is given next where the second subscript refers to the player. Also, $\sum_i \beta_{i,j} = 0$ for $j = 1, 2, 3$.

$$(6) \quad \begin{aligned} \mathcal{G}^N : \text{Front} &= \begin{array}{|c|c|c|} \hline \eta_{1,1} & \eta_{1,2} & \eta_{1,3} \\ \hline -\eta_{1,1} & \eta_{2,2} & \eta_{3,3} \\ \hline \eta_{2,1} & -\eta_{1,2} & \eta_{2,3} \\ \hline -\eta_{2,1} & -\eta_{2,2} & \eta_{4,3} \\ \hline \end{array}, \\ \text{Back} &= \begin{array}{|c|c|c|} \hline \eta_{3,1} & \eta_{3,2} & -\eta_{1,3} \\ \hline -\eta_{3,1} & \eta_{4,2} & -\eta_{3,3} \\ \hline \eta_{4,1} & -\eta_{3,2} & -\eta_{2,3} \\ \hline -\eta_{4,1} & -\eta_{4,2} & -\eta_{4,3} \\ \hline \end{array}, \\ \mathcal{G}^B : \text{Front} &= \begin{array}{|c|c|c|} \hline \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ \hline \beta_{1,1} & \beta_{2,2} & \beta_{3,3} \\ \hline \beta_{2,1} & \beta_{1,2} & \beta_{2,3} \\ \hline \beta_{2,1} & \beta_{2,2} & \beta_{4,3} \\ \hline \end{array}, \\ \text{Back} &= \begin{array}{|c|c|c|} \hline \beta_{3,1} & \beta_{3,2} & \beta_{1,3} \\ \hline \beta_{3,1} & \beta_{4,2} & \beta_{3,3} \\ \hline \beta_{4,1} & \beta_{3,2} & \beta_{2,3} \\ \hline \beta_{4,1} & \beta_{4,2} & \beta_{4,3} \\ \hline \end{array}, \end{aligned}$$

where each cell entry for the front and the back matrices of \mathcal{G}^K is $(\kappa_1 \kappa_2 \kappa_3)$ and κ_j is the average of the \mathcal{G} entries for the j^{th} player. For a 2×2 game, only use the "Front" matrix and drop the third player's entries.

1.2. Advantages. Computing the Nash \mathcal{G}^N component can be quickly done by using nothing more than elementary arithmetic. Of importance, this \mathcal{G}^N component identifies all possible aspects of a game's Nash structure.

To illustrate how this decomposition provides added value, a normal Nash analysis for 2×2 games involves computations with eight variables. But with \mathcal{G}^N , only the four defining $\eta_{1,1}, \eta_{2,1}, \eta_{1,2}, \eta_{2,2}$ variables are needed. More generally, the space of games for n agents, where each player has two strategies, is $\mathbb{G} = \mathbb{R}^{n2^n}$, so a Nash analysis typically involves all $n2^n$ terms. But with the decomposition, only the $n2^{n-1}$ variables $\{\eta_{i,j}\}$ from \mathcal{G}^N – half as many – are needed. Of importance for what is developed here, the $n(2^{n-1} - 1)$ $\{\beta_{i,j}\}$ variables that define the behavioral component \mathcal{G}^B completely determine all other aspects of the game. What simplifies the analysis is the separation of the η and β variables.

The computational simplicity of determining \mathcal{G}^N provides an easy way to identify all pure Nash equilibria. This follows from the Eq. 6 representation for \mathcal{G}^N :

After all other players select their strategies, the remaining player, say the j^{th} , has two options with payoffs $-|\eta_{i,j}|$ and $|\eta_{i,j}|$. To be a Nash equilibrium, this player must select the larger $|\eta_{i,j}|$ choice. This leads to the first Thm. 1.1 statement, where the “strict” modifier just eliminates $\eta_{i,j} = 0$ terms.

THEOREM 1.1. *For games \mathcal{G} involving $n \geq 2$ players where each has two strategies, the following are true:*

- (1) *For game \mathcal{G} , a strict pure Nash equilibrium occurs if and only if all of the entries in the identified \mathcal{G}^N cell are positive.*
- (2) *A game \mathcal{G} can be constructed to have k pure Nash equilibria where k is any integer satisfying $0 \leq k \leq 2^{n-1}$.*
- (3) *Mixed strategies are completely determined by the $n2^{n-1}$ variables $\{\eta_{i,j}\}$ that define \mathcal{G}^N .*

The third statement follows immediately from the structure of the \mathcal{G}^B and \mathcal{G}^K terms; neither provide any strategic opportunity for any player. To illustrate with a 2×2 game

$$(7) \quad \begin{array}{|c|c|c|c|} \hline 5 & 5 & 1 & 1 \\ \hline 3 & 0 & 3 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & -1 & -2 \\ \hline -1 & -1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 1 & -1 & 1 \\ \hline 1 & -1 & -1 & -1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline 3 & 2 & 3 & 2 \\ \hline 3 & 2 & 3 & 2 \\ \hline \end{array},$$

if the column player plays L-R, respectively, with probability q and $(1 - q)$, then the difference between the expected values for the row player of playing T and B for the \mathcal{G}^N part of the game (the first bimatrix to the right of the equal sign) is

$$(8) \quad E(T) - E(B) = [q - (1 - q)] - [-q + (1 - q)] = 2q - 2(1 - q).$$

If the $E(T) - E(B)$ value is positive, the player should play T, if it is negative, the player should play B. In particular, this expression leads to the column player’s mixed strategy of $q = \frac{1}{2}$.

Now consider what happens with the original Eq. 7 game (the game on the left); this difference in $E(T) - E(B)$ expected values is

$$(9) \quad E(T) - E(B) = [5q + (1 - q)] - [3q + 3(1 - q)] = 2q - 2(1 - q) + [(4 - 4)q + (2 - 2)(1 - q)] = 2q - 2(1 - q).$$

The fact Eqs. 8 and 9 result in the same $E(T) - E(B) = 2q - 2(1 - q)$ equation is not a coincidence. The reason is demonstrated with the bracketed term in the penultimate equality of Eq. 9; it consists of terms from the \mathcal{G}^B and \mathcal{G}^K components. By construction, these components always have the same value (for the row player) for q^{th} column and for the $(1 - q)^{th}$ column (see the above definitions for \mathcal{G}^B and \mathcal{G}^K), which means that these terms in the bracket must always cancel. What remains in Eq. 9 are the $\eta_{i,j}$ values that define Eq. 8.

The same structure holds for any number of players. With three players and Eq. 6 where the column player plays L with probability q and the third player plays F with probability r , the qr coefficient in the row’s player’s difference between the expected values of playing T and B is

$$[\eta_{1,1} + \beta_{1,1}] - [-\eta_{1,1} + \beta_{1,1}] = 2\eta_{1,1}$$

showing again (as asserted after Eq. 3) that the \mathcal{G}^B and \mathcal{G}^K components play no role in determining Nash strategies. This leads to the third assertion of the theorem.

Before outlining a proof for the second statement, it is worth indicating how we will use this construction. The outlined proof for the second assertion makes it easy

to create a \mathcal{G}^N with any desired Nash structure. This permits creating a wide class of games where each game has the same Nash structure, but different games exhibit different cooperative behavior in repeated settings. To do so, design a desired \mathcal{G}^N component that will be common for all of the games. Next, select appropriate \mathcal{G}^B components, and add them (by using Eq. 5) to obtain $\mathcal{G} = \mathcal{G}^N + \mathcal{G}^B$. (This is how we created the Eq. 10 games in the next section.)

To illustrate, it can be difficult to create a class of four-player games where each game has the same $2^{4-1} = 8$ pure Nash equilibria, but different games have different cooperative behavior in a repeated setting. With the description given next, the \mathcal{G}^N component is easily created; what remains is to find appropriate \mathcal{G}^B components.

The second statement can be proved with an obvious induction argument. To illustrate, we will construct four-player \mathcal{G}^N that has $2^{4-1} = 8$ pure Nash equilibria; as it will become clear, these equilibria must have a diametric positioning in \mathcal{G}^N .

To start, TL is a Nash equilibrium for a two player \mathcal{G}^N game only if its two entries are positive. According to the decomposition, this choice requires a negative entry in the row player's BL cell and the column player's TR cell. With these negative values, BR is the only possible cell that also could be a Nash equilibrium. To make BR an equilibrium, select its two entries to be positive. In turn, this choice determines all remaining entries of this \mathcal{G}^N .

To build a three player \mathcal{G}^N game, start with the above TL and BR structure of the two person game in the Front set of matrices. For TLF and BRF to be strict Nash equilibria, it is necessary to assign a positive entry for player 3 in these \mathcal{G}^N cells. This choice and the structure of $\eta_{i,3}$ properties require Player 3 to have a negative entry in TLBa and BRBa; the entries for Players 1 and 2 for this Back choice are free to be selected. The only way to obtain two more equilibria (which must be in the Back matrix) is to place positive entries for all players in the remaining two \mathcal{G}^N Back cells (that is, TRBa and BLBa). Once these choices are made, the entries for all cells are determined. Only the selected ones have all positive entries, so the four equilibria are located at TLF, BRF, BLBa, and TRBa.

A four player game is given by two sets, I and II, of "Front-Back" matrices. Namely, it consists of two sets of matrices with the Eq. 4 form except that each cell lists the payoffs for each of the four players. To have four equilibria in set I, put a positive value for Player 4 in each of TLF_I, BRF_I, BLBa_I, and TRBa_I. This forces a negative value for Player 4 in each of the TLF_{II}, BRF_{II}, BLBa_{II}, and TRBa_{II} cells. Select positive $\eta_{i,j}$ values for the remaining four set II cells; this determines the remaining entries and creates a \mathcal{G}^N with eight pure Nash equilibria. More generally, moving from $(n - 1)$ to n players doubles the maximum number of pure Nash equilibria.

The mixed Nash equilibria for the above four-person game are found in the standard way with these 32 $\eta_{i,j}$ variables. To construct a wide class of games with this same Nash structure but potentially different equilibria for repeated games, add appropriately selected \mathcal{G}^B terms to the constructed \mathcal{G}^N .

1.3. Three examples. To introduce the types of games, strategies, and unusual conclusions that are analyzed in this paper, consider the three Eq. 10 games.

$$\begin{aligned}
 (10) \quad \mathcal{G}_2 \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 4 & 4 & 4 \\ \hline 4 & 4 & 5 & 8 & 8 & 7 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 5 & 5 & 4 & 7 & 9 & 8 \\ \hline 9 & 7 & 9 & 11 & 11 & 11 \\ \hline \end{array} \\
 \mathcal{G}_3 \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline 7 & 7 & 7 & 5 & 11 & 5 \\ \hline 11 & 5 & 4 & 9 & 9 & 0 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 4 & 11 & 0 & 8 & 9 \\ \hline 8 & 0 & 8 & 4 & 4 & 4 \\ \hline \end{array} \\
 \mathcal{G}_4 \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline 7 & 4 & 5 & 5 & 8 & 7 \\ \hline 11 & 7 & 4 & 9 & 11 & 0 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 7 & 9 & 0 & 11 & 11 \\ \hline 8 & 0 & 8 & 4 & 4 & 4 \\ \hline \end{array}
 \end{aligned}$$

To connect these three games with the discussion of the previous section, they were designed by first selecting the common \mathcal{G}^N given in Eq. 11, which has the dominant BRBa strategy. All differences among these three Eq. 10 games, then, completely reflect the choices for the \mathcal{G}^B component as described in Eq. 12. Whatever choice is made for \mathcal{G}^B , the dominant Nash strategy for each of these games remains BRBa.

$$(11) \quad \mathcal{G}_i^N \quad \text{Front} = \begin{array}{|c|c|c|c|c|c|} \hline -2 & -2 & -2 & -2 & 2 & -2 \\ \hline 2 & -2 & -2 & 2 & 2 & -2 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline -2 & -2 & 2 & -2 & 2 & 2 \\ \hline 2 & -2 & 2 & 2 & 2 & 2 \\ \hline \end{array}$$

$$(12) \quad \begin{aligned}
 \mathcal{G}_2^B \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline -4 & -4 & -4 & 0 & -4 & 0 \\ \hline -4 & 0 & 1 & 0 & 0 & 3 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & -4 & 3 & 1 & 0 \\ \hline 1 & 3 & 1 & 3 & 3 & 3 \\ \hline \end{array} \\
 \mathcal{G}_3^B \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline 3 & 3 & 3 & 1 & 3 & 1 \\ \hline 3 & 1 & 0 & 1 & 1 & -4 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 3 & -4 & 0 & 1 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline \end{array} \\
 \mathcal{G}_4^B \quad \text{Front} &= \begin{array}{|c|c|c|c|c|c|} \hline 3 & 0 & 1 & 1 & 0 & 3 \\ \hline 3 & 2 & 0 & 1 & 2 & -4 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 2 & 1 & -4 & 2 & 3 \\ \hline 0 & -4 & 0 & -4 & -4 & -4 \\ \hline \end{array}
 \end{aligned}$$

$$(13) \quad \mathcal{G}_i^K \quad \text{Front} = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline \end{array}, \quad \text{Back} = \begin{array}{|c|c|c|c|c|c|} \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 6 \\ \hline \end{array}$$

A first objective is to determine which states for each game can be sustained in a repeated setting. For \mathcal{G}_2 , the answer is immediate: Each player has a dominant strategy as given by BRBa. As BRBa also yields the game’s Pareto superior outcome, by playing the dominant strategy, each player receives the game’s highest possible payoff.

Results for \mathcal{G}_3 are not as obvious. Each player has a dominant BRBa strategy, but the game’s payoff of “4” for each player is Pareto inferior to the TLF, TRF, TLBa, and BLF outcomes. It is reasonable to wonder whether any of these four choices can be sustained should each player adopt, say, a grim-trigger strategy. What makes this question of particular interest is that the TLF entry, which is strictly preferred to BRBa, creates a three-person Prisoners’ Dilemma. A natural question is to determine whether cooperative behavior of playing T, L, and F can be sustained with grim trigger or tit-for-tat. As we show, if the discount factor satisfies $\delta > \frac{4}{7}$, then grim trigger does support TLF. Even stronger, with $\frac{4}{7} < \delta \leq \frac{4}{5}$, TLF is the *only outcome* that grim trigger can sustain. But if $\delta > \frac{4}{5}$, then TRF can be sustained as well. The next question is whether larger δ values could support BLF or TLBa: They cannot; no other state is inducible under grim trigger.

Notice how this conclusion identifies an advantage for Player 2 over the other players. This is because the two sustainable grim trigger options for $\delta > \frac{4}{5}$ are

TLF and TRF: Only Player 2 can select between them. Consequently, Player 2 can receive the personally higher TRF payoff at the expense of the other two players.

This result supports one of our themes: *Features, properties, and conclusions that are derived from two-person games need not transfer to three-person games.* The particular feature illustrated here is that, with a two-person Prisoners' Dilemma game, if cooperation is ensured for a δ_1 value, then cooperation is ensured for $\delta \geq \delta_1$. But, this property *need not* transfer to three-person Prisoners' Dilemma games. Instead, and as demonstrated with \mathcal{G}_3 , even if a particular δ value (here, $\frac{4}{7} < \delta \leq \frac{4}{5}$) can sustain cooperation among all three players (to attain TLF), this cooperation can disappear with a larger δ value (here $\delta > \frac{4}{5}$)! This larger δ value makes it to the second player's advantage to defect from the cooperative TLF strategy to obtain the personally preferred outcome from TRF; this defection is at the expense of the other two players. Of interest is how classes of games with this feature can be identified by using the decomposition and analysis developed below.

Turning to the last game, while the dominant strategy outcome for \mathcal{G}_4 is BRBa, the TLF, BLF, TRF, and TLBa choices have Pareto preferred outcomes (where TLF again generates a three-person Prisoners' Dilemma). A reason \mathcal{G}_4 appears to be the most complicated game is that it does not have a clear optimal state. Indeed, this complexity is reflected by the number of calculations that typically need to be performed in a standard analysis. However, by using the method outlined below, it can be quickly shown that, for any $\delta \in (0, 1)$, it is impossible to induce a cooperative state with a grim-trigger strategy. ("Grim trigger" is emphasized only to simplify computations.)

These three games illustrate different interesting features that are strictly due to behavioral \mathcal{G}^B terms in Eq. 12. This must be the case because games $\mathcal{G}_2, \mathcal{G}_3$ and \mathcal{G}_4 are strategically equivalent in the sense that $\mathcal{G}_2^N = \mathcal{G}_3^N = \mathcal{G}_4^N$. This equality means that all possible differences among these games in achieving cooperation are strictly caused by differences in the games' behavioral \mathcal{G}^B components.

To describe the \mathcal{G}^B differences, the \mathcal{G}_2^B choice has its Pareto superior entry agreeing with the location of the \mathcal{G}_2^N Pareto superior entry; this agreement ensures that \mathcal{G}_2 has a dominant strategy that also is the Pareto superior outcome. In contrast, the \mathcal{G}_3^B choice has its Pareto superior entry located at the position of the \mathcal{G}_3^N Pareto inferior entry, and diametrically opposite the \mathcal{G}_3^N Pareto superior term; the features of this \mathcal{G}_3^B choice are what allow only two sustainable grim trigger cooperative outcomes. Finally, the choice of \mathcal{G}_4^B was selected so that its features prohibit any grim trigger sustainable cooperative outcome.

Another one of our other themes is captured by the similarity of games \mathcal{G}_3 and \mathcal{G}_4 : Player 1's entries in both games are identical, and there are only minor differences in the entries for the other two players. The point to be made is that while the game matrices are similar, the cooperative structures with repeated play differ significantly in that \mathcal{G}_3 can ensure cooperation, but \mathcal{G}_4 cannot.

As this example makes clear, changes in \mathcal{G}^B , even small changes, can alter what kinds of cooperation are possible. Indeed, the two games could have been made to be much closer in appearance, but we opted to use choices where all terms are integers. Stated in mathematical terms, by identifying the boundary of \mathcal{G}^B choices that separates different repeated game behaviors, the entries for the two games can be selected to be arbitrarily close. As such and as asserted in our

introductory comments, even a small perturbation of a given, specified game can result in radically different game-theoretic behavior.

It is reasonable to wonder whether these conclusions would change by replacing the grim-trigger with the tit-for-tat strategy. They can, but as shown for the Eq. 10 games, tit-for-tat strategies produce identical results.

2. Sustaining Cooperation

Our next goal is to show how a game's \mathcal{G}^B behavioral structure can change the game's cooperative properties. To prevent this discussion from becoming obscured with technical computations, we consider only the commonly used grim-trigger and tit-for-tat strategies. (Interested readers may wish to develop the properties of other strategies.) Also, our intent differs from the usual objective; it is to determine what happens in a repeated setting with all possible *non-Nash outcomes*. In this more general way, our analysis includes as a special case the typical choice of exploring whether a Pareto optimal outcome can be sustained.

The Eq. 10 games illustrate a feature of our more general perspective: While \mathcal{G}_2 and \mathcal{G}_3 have clear candidates for the cooperation state, BRBa for \mathcal{G}_2 and TLF for \mathcal{G}_3 , it is not clear what should be selected for \mathcal{G}_4 . With \mathcal{G}_4 , for instance, the three states of TRBa, TRF, and BLF are Pareto equivalent. But by characterizing all sustainable outcomes in any game (independent of their Pareto properties), this selection problem can be ignored; the choice becomes a special case of checking what positive results emerge from the analysis. Namely, our more general approach of examining all non-Nash outcomes sheds light on differing ways to define what is an optimal outcome.

By characterizing which states are inducible in a repeated setting, we also identify the role played by \mathcal{G}^B in seeking cooperative outcomes and how the choice of \mathcal{G}^B can change the role of \mathcal{G}^N in the analysis. To simplify the calculations, the punishment state is given by a pure-strategy outcome. An interesting feature is that the punishment state need not be an equilibrium of the single-shot game. This flexibility allows for a minmax strategy against Player i to be i 's punishment.

2.1. The effect of the \mathcal{G}^B component. It is instructive to observe how the \mathcal{G}^B structure changes the role played by \mathcal{G}^N for a given game \mathcal{G} . Should \mathcal{G}^B have a minimal impact on \mathcal{G}^N (e.g., the magnitudes of the \mathcal{G}^B entries are small in size (relative to \mathcal{G}^N) or \mathcal{G}^B supports the \mathcal{G}^N structure by enhancing a Nash equilibrium (as true with \mathcal{G}_2^B), then \mathcal{G}^B plays a minimal role in the \mathcal{G} game analysis. Here, most of what is interesting in the game can be attributed to the \mathcal{G}^N component; this is where what individuals obtain (the Nash equilibria) is determined by their own strategic actions. But a stronger \mathcal{G}^B term, such as with \mathcal{G}_3^B , provides an inducement for the players to try to obtain a Pareto preferred outcome. Namely, ways must be found to extract the benefits that are provided by the \mathcal{G}^B component: Doing so requires a level of cooperation among the players.

Our goal is to determine what it takes to ensure cooperation that will, for *any* reason, support *any* specified non-Nash outcome for *any* game. For the desired cooperation to be feasible, each player must receive a better outcome with the targeted outcome than what the player could get from at least one other setting. No assumption is made about the structure of this comparison, so this analysis includes, as a special case, the Prisoners' Dilemma (where each player's comparison entry is determined by the dominant strategy). In particular, the targeted outcome

need not be Pareto superior to the various Nash equilibria (Thm. 1.1); all that is required is that the targeted outcome is better than *some other* outcome. Thus this discussion applies to a wide selection of games.

A feature described below (Sect. 2.2) is that the effort needed to support a non-Nash outcome always is accompanied with a temptation for certain players to renege from cooperating. (This feature holds the Prisoner's Dilemma, but it is not obvious that it is true in general.) A reneging player could destroy the designated cooperative outcome, so, if the targeted outcome is desired by the other players, they must adopt some form of collective action to enforce cooperation. A natural approach is to make it expensive to renege. Doing so may not be possible in a one-shot game, but the added opportunities offered by a repeated game may make it an option. Strategies such as the grim trigger and tit-for-tat share this objective; they differ in their efficiency and effectiveness.

To make "reneging" costly, a game must have a sufficiently distasteful aspect that can be converted into an enforcement tool; the goal is to have the non-cooperative player suffer this distasteful punishment. The question is whether a game always provides such enforcement tools. Surprisingly, it always does; this enforcing tool comes from the structure of \mathcal{G}^N !

In summary, for any given game \mathcal{G} , the role played by \mathcal{G}^N is influenced by the \mathcal{G}^B structure. When \mathcal{G}^N is the dominating component of the game, it enjoys a positive, rewarding image of determining what should happen. But with changes in a game due to \mathcal{G}^B inducements, the image of \mathcal{G}^N now changes from describing rewards to that of providing a means for punishment for the lack of cooperation.

2.2. Cooperation. Once each of $(n - 1)$ players adopt a strategy, the remaining player, say Player 1, is left with the two choices centered about their $\beta_{i,1}$ average. Remember from the decomposition that the $\beta_{i,1}$ averaged value is the same for either choice. As such, the player selects between a larger $|\eta_{i,1}| + \beta_{i,1}$ or smaller $-|\eta_{i,1}| + \beta_{i,1}$, where the larger $|\eta_{i,1}| + \beta_{i,1}$ payoff strategically dominates.

Now consider what is required to encourage a player to cooperatively support a specified non-Nash outcome. It follows from the definition of being non-Nash that for some player, say Player 1, to cooperate requires selecting the strategy offering the poorer $-|\eta_{i,1}| + \beta_{i,1}$ rather than the more tempting $|\eta_{i,1}| + \beta_{i,1}$; by selecting the poorer value when a larger one is available reflects the above described guaranteed temptation. This comment applies to any non-Nash outcome, so it generalizes the usual discussions associated with the Prisoners' Dilemma.

Why would Player 1 be willing to cooperate if doing so involves the personal expense of adopting a poorer choice? One reason is that the poorer $-|\eta_{i,1}| + \beta_{i,1}$ payoff must be preferred to some other outcome where Player 1 receives $|\eta_{j,1}| + \beta_{j,1}$; i.e., if Player 1 does not cooperate, the other players might be able to force Player 1's future choices to be made between the j^{th} two entries, rather than the i^{th} two entries. For this property to hold, the payoffs must satisfy

$$(14) \quad |\eta_{i,1}| + \beta_{i,1} > -|\eta_{i,1}| + \beta_{i,1} > |\eta_{j,1}| + \beta_{j,1}.$$

There are no opportunities to ensure cooperation in a one shot game. But in a repeated game, the $(n - 1)$ other players can select penalty strategies to force Player 1 to choose between $|\eta_{j,1}| + \beta_{j,1}$ and $-|\eta_{j,1}| + \beta_{j,1}$. The main differences between grim trigger, tit-for-tat, and various modifications are the frequency and reasons to penalize; our interest is to identify when cooperation can be achieved.

The answer depends on the effect of the future penalties on Player 1 as determined by the player's discount factor $\delta \in (0, 1)$.

THEOREM 2.1. *In an $n \geq 2$ player game, suppose with a targeted, non-Nash cooperative outcome, Player 1, with discount rate $\delta \in (0, 1)$, must select a strategy yielding the smaller of $\{-|\eta_{i,1}| + \beta_{i,1}, |\eta_{i,1}| + \beta_{i,1}\}$ as identified in Eq. 14. To enforce this cooperative action, the other $(n - 1)$ players use the grim trigger option that would force Player 1 to select between $|\eta_{j,1}| + \beta_{j,1}$ and $-|\eta_{j,1}| + \beta_{j,1}$. It is in Player 1's interest to cooperate if*

$$(15) \quad \left(1 - \frac{2}{\delta}\right) |\eta_{i,1}| + \beta_{i,1} > |\eta_{j,1}| + \beta_{j,1}$$

If the other players adopt a tit-for-tat strategy, then it is in Player 1's interest to cooperate if

$$(16) \quad \left(-1 - \frac{2}{\delta}\right) |\eta_{i,1}| + \beta_{i,1} > -|\eta_{j,1}| + \beta_{j,1}$$

PROOF. Suppose Player 1 is facing opponents who have implemented a grim trigger in which he can play Top (Cooperate) for a payoff of $-|\eta_{i,1}| + \beta_{i,1}$ ad infinitum, or play Bottom (Defect) for a one-time payoff of $|\eta_{i,1}| + \beta_{i,1}$. But if our player did not cooperate, the grim trigger ensures that the subsequent payoffs are $|\eta_{j,1}| + \beta_{j,1}$ ad infinitum. This means that Player 1 will cooperate if

$$(17) \quad \begin{aligned} \sum_{t=1}^{\infty} \delta^{t-1} (-|\eta_{i,1}| + \beta_{i,1}) &> (|\eta_{i,1}| + \beta_{i,1}) + \sum_{t=2}^{\infty} \delta^{t-1} (|\eta_{j,1}| + \beta_{j,1}), \\ \frac{-|\eta_{i,1}| + \beta_{i,1}}{1 - \delta} &> |\eta_{i,1}| + \beta_{i,1} + \delta \left(\frac{|\eta_{j,1}| + \beta_{j,1}}{1 - \delta} \right), \\ -|\eta_{i,1}| + \beta_{i,1} - (1 - \delta)(|\eta_{i,1}| + \beta_{i,1}) &> \delta(|\eta_{j,1}| + \beta_{j,1}), \\ (\delta - 2)|\eta_{i,1}| + \delta\beta_{i,1} &> \delta(|\eta_{j,1}| + \beta_{j,1}) \end{aligned}$$

After some algebraic computations, Eq. 15 is obtained.

If Player 1 is faced with opponents who are playing tit-for-tat, then the grim-trigger in Eq. 15 must hold or Player 1 will never cooperate, as he retains the ability to defect ad infinitum against. However, there is also the possibility of first cooperation, then defection, then cooperation, etc. In order for cooperation to hold in this case, it is also needed that

$$\begin{aligned} \sum_{t=1}^{\infty} \delta^{t-1} (-|\eta_{i,1}| + \beta_{i,1}) &> |\eta_{i,1}| + \beta_{i,1} + \delta (-|\eta_{j,1}| + \beta_{j,1}) \\ &\quad + \delta^2 (|\eta_{i,1}| + \beta_{i,1}) + \delta^3 (-|\eta_{j,1}| + \beta_{j,1}) + \dots \\ \sum_{t=1}^{\infty} \delta^{t-1} (-|\eta_{i,1}| + \beta_{i,1}) &> \sum_{t=1}^{\infty} \delta^{2(t-1)} (|\eta_{i,1}| + \beta_{i,1}) + \delta \sum_{t=1}^{\infty} \delta^{2(t-1)} (-|\eta_{j,1}| + \beta_{j,1}) \\ \frac{-|\eta_{i,1}| + \beta_{i,1}}{1 - \delta} &> \frac{|\eta_{i,1}| + \beta_{i,1}}{1 - \delta^2} + \delta \left(\frac{-|\eta_{j,1}| + \beta_{j,1}}{1 - \delta^2} \right) \\ \left(\frac{1 + \delta}{1 + \delta} \right) \frac{-|\eta_{i,1}| + \beta_{i,1}}{1 - \delta} &> \frac{|\eta_{i,1}| + \beta_{i,1}}{1 - \delta^2} + \delta \left(\frac{-|\eta_{j,1}| + \beta_{j,1}}{1 - \delta^2} \right) \\ (1 + \delta)(-|\eta_{i,1}| + \beta_{i,1}) &> |\eta_{i,1}| + \beta_{i,1} + \delta (-|\eta_{j,1}| + \beta_{j,1}) \\ (-\delta - 2)|\eta_{i,1}| + \delta\beta_{i,1} &> \delta (-|\eta_{j,1}| + \beta_{j,1}) \end{aligned}$$

This inequality is equivalent to Eq. 16. □

Equations 15, 16 completely characterize all pairs of states (the targeted and the penalty states) in which cooperation is inducible against strategic interests with either a grim trigger or a tit-for-tat strategy, respectively. Also, for $\delta \in (0, 1)$, the left hand side of these equations is unbounded below whenever $\eta_{i,1} \neq 0$, so all statements must be qualified with suitable conditions on the discount factor.

As Eqs. 15 and 16 show, equilibrium outcomes in repeated game are affected by $\beta_{i,1}, \beta_{j,1}$ values; this information is totally ignored in the single-shot case. This is because $\beta_{i,j}$ values are irrelevant when computing a single-shot Nash equilibrium. (See the discussion following Thm. 1.1.) On the other hand and as demonstrated in these two equations, $\beta_{i,j}$ values have a large impact on whether or not cooperation can be induced in repeated games. Also note that the phrase “for δ large enough” could be substituted with “for non-strategic interests large enough” (i.e., “for sufficiently large $\beta_{i,1}$ values”), or “for strategic factors small enough” (i.e., “for sufficiently small $\eta_{i,1}$ values”), when describing results on cooperative outcomes.

This analysis captures one of our goals, which is to indicate how the game structure – the level of inducement given by \mathcal{G}^B – can be modified to attain and sustain cooperation. Furthermore, these equations also identify the type of information that parameter δ measures; as δ is a coefficient only for the $\eta_{i,j}$ terms, non-strategic interests (i.e., \mathcal{G}^B terms) can be altered without affecting the standard requirements of a “suitable δ .”

2.3. An application. To support our assertions about the three Eq. 10 games by using Thm. 2.1, notice that when $|\eta_{i,1}| = |\eta_{j,1}|$, the inequalities in Eqs. 15 and 16 are identical. It is only when $|\eta_{i,1}| > |\eta_{j,1}|$ that the tit-for-tat inequality imposes a greater restriction. If $|\eta_{i,1}| < |\eta_{j,1}|$, then the grim trigger inequality is stricter; it must also hold for the tit-for-tat strategy to sustain cooperation. This demonstrates that the strength of the inducement strategy is a function of the \mathcal{G}^N strategic structure of the game; in some games, a tit-for-tat strategy is not stronger than a grim-trigger strategy. For the games in Eq. 10, the two are equivalent, which is clear with the Eqs. 11, 12 decompositions.

To determine which states are equilibria in a repeated setting for games \mathcal{G}_3 and \mathcal{G}_4 , first note that the punishment for defection can be assumed to be BRBa as this is both the minmax against each player and the Nash equilibrium with the lowest payoff. Because $\eta_{i,j} = -2$ for all i, j , Eq. 15 gives

$$(18) \quad 2 - \frac{4}{\delta} + \beta_{i,j} > 2 - 4 \Rightarrow \beta_{i,j} > -4 + \frac{4}{\delta}$$

This inequality makes it easy to determine the equilibria states; just check whether the $\beta_{i,j}$ values satisfy this inequality. In particular, if $\delta > \frac{4}{7}$, then a state is sustainable if $\beta_{i,j} \geq 3$ for every Player j . This value holds only for the TLF outcome in \mathcal{G}_3 . Increasing δ values make it easier to sustain cooperation; indeed, for $\delta > \frac{4}{5}$, a state is sustainable if $\beta_{i,j} \geq 1$. This means that TRF in \mathcal{G}_3 also is an equilibrium. The limiting case as $\delta \rightarrow 1$ is $\beta_{i,j} > 0$. No other outcomes in either \mathcal{G}_3 or \mathcal{G}_4 has each player with $\beta_{i,j} > 0$, so there are no more equilibria. So, searching for *all* equilibria in a repeated setting can simplify the computations. Namely, by developing a characterization of all states that are sustainable in a repeated setting, and doing this in terms of the structure of the game, the amount of calculating can be substantially reduced.

These examples highlight the importance of the $\beta_{i,j}$ values in determining the repeated game equilibria. This relationship is captured by the following theorem where $\mathcal{G}_i \sim_N \mathcal{G}_j$ means that $\mathcal{G}_i^N = \mathcal{G}_j^N$. Also recall that \mathbb{G} is the space of all games; in Thm. 2.2, \mathbb{G} can be the set of all 2×2 games or the set of all $2 \times 2 \times 2$ games, but the conclusion holds even if \mathbb{G} is the set of all $2 \times \dots \times 2$ games with n agents.

THEOREM 2.2. *For any $\mathcal{G} \in \mathbb{G}$, any $\delta \in (0, 1)$, any choice of cooperation outcome, and any different choice of punishment outcome, there exists a game $\mathcal{G}' \in \mathbb{G}$ such that $\mathcal{G}' \sim_N \mathcal{G}$ and the cooperation outcome can be sustained in \mathcal{G}' with either a tit-for-tat or grim-trigger strategy. Furthermore, the payoffs in \mathcal{G}' can be restricted to any interval in \mathbb{R} .*

PROOF. For any given \mathcal{G} , the $\beta_{i,j}$ values can be adjusted freely. Eqs. 15 and 16, as well as the Nash component \mathcal{G}^N , are invariant under both positive scalar multiplication and addition by a constant to all of Player i 's payoffs. \square

The significance of Thm. 2.2 is to demonstrate the importance of the actual magnitude of values that are in repeated games. Often, inspiration for a repeated game comes from a story about the Nash equilibrium structure of the single-shot game. But as this theorem shows, information from a single shot analysis is insufficient to determine what might happen in a repeated game analysis. Furthermore, Thm. 2.2 shows that there can be hidden difficulties in picking a “representative game” from an equivalence class of games, such as a reduction of a game to its ordinal information. An alternative approach is to use the Eq. 15, 16 inequalities to identify *classes* of relevant games in which a desired outcome holds.

2.4. Encouraging cooperation. In the introductory comments it was mentioned that the behavioral structure of a game \mathcal{G}^B can be altered to encourage cooperation in the repeated setting without affecting the strategic structure \mathcal{G}^N . Information about how to do so follows directly from Eqs. 15 and 16.

The approach is immediate; increasing the $\beta_{i,1} - \beta_{j,1}$ value improves the incentives for Player 1 to cooperate. For instance, as \mathcal{G}_4 does not have a sustainable equilibria in a repeated game, can \mathcal{G}_4 be altered without affecting its strategic structure so that the resulting repeated game now has such an equilibrium?

To illustrate how to do so, select a non-Nash entry from \mathcal{G}_4 ; any one will do. Choose, for instance, BLF where $\beta_{2,2} = 2$ and $\beta_{3,3} = 0$. (For a reminder of the $\beta_{i,j}$ notation, see Eq. 6.) With the given Nash structure, a way to make this equilibrium sustainable in a repeated setting is to increase both β values to at least 3 (Eq. 18). But changing $\beta_{2,2}$ from 2 to 3 changes the second player's \mathcal{G}^B entries in BLF and in BRF, and changing $\beta_{3,3}$ from 0 to 3 changes the third player's \mathcal{G}^B entries in BLF and in BLBa. In other words, these inducements must be of the kind that are available whether the player cooperates, or caves to temptation.

The nature of these β terms depends upon what is being modeled. If, for instance, the payoffs related to the \mathcal{G}_4^N component represent what can happen with military action, the $\beta_{i,j}$ entries may represent added benefits for the player under peace. In fact, in practice, while the strategic \mathcal{G}^N aspects of a game may remain fixed, negotiations typically involve changing a game – changing the payoffs – to induce players to cooperate. These changes, then, are reflected by what it takes to alter the \mathcal{G}^B portion of a game to attain cooperation.

A second issue is to determine what is needed to encourage cooperation with players who have less regard for the future as captured by a smaller δ value. With

\mathcal{G}_3 , for instance, suppose $\delta = \frac{1}{2}$; how could \mathcal{G}_3 be altered to make TLF a sustainable equilibrium in the repeated game? Again, the answer comes from Eq. 18; find behavioral inducements (as represented by the β values) to convert the current \mathcal{G}_3^B TLF values from all 3's to perhaps all 4.1's.

A third issue follows from the reality that $n > 2$ player games admit added cooperative equilibria. As demonstrated with \mathcal{G}_3 , this wealth of equilibria can introduce new difficulties. Here, an increasing δ value, where the players more strongly value the future, jeopardizes the cooperative TLF solution by introducing the option of the TRF equilibria. A way to counter such behavior is to change one of the β entries in the TRF cell of \mathcal{G}_3^B to be zero or negative.

In fact, the Eq. 10 games were designed in precisely this manner. After obtaining Eq. 18 from the Nash structure of a game, the \mathcal{G}^B components can be designed to illustrate whatever features are desired.

3. Cooperation Inducement

Aside from payoff specification, a difficulty in analyzing repeated games comes from having additional players. As it was pointed out in Sect. 1.3 when discussing properties of \mathcal{G}_3 , results obtained from simplifying a situation to two players do not necessarily reflect what occurs in a more realistic setting with added players.

In [1], Brams and Kilgour analyze the cooperation inducements in a 2-player repeated setting. They show that by giving one player the common knowledge ability to receive a noisy signal of the opponent's strategy before play, a probabilistic tit-for-tat strategy can induce a cooperative outcome in a significant proportion of games. A natural question is whether this kind of conclusion extends to 3 players.

In a 2-player setting, giving Player 1 the common knowledge ability to observe the opponent's strategy decreases, but does not eliminate, the incentive for defection: if Player 2 defects, the likelihood of receiving a higher one-time payoff is no longer certain, but the punishment in the following round remains. To see this, consider the following simplified 2-player game consisting only of Player 2's payoffs

$$(19) \quad \mathcal{G}_4 = \begin{array}{|c|c|} \hline -|\eta_{1,2}| & |\eta_{1,2}| \\ \hline -|\eta_{2,2}| & |\eta_{2,2}| \\ \hline \end{array} + \begin{array}{|c|c|} \hline \beta_{1,2} & \beta_{1,2} \\ \hline \beta_{2,2} & \beta_{2,2} \\ \hline \end{array}.$$

With this structure, Player 2 has the dominant strategy of R. So, assume that the cooperative strategy for the game is TL. Without a detection mechanism, Player 2 will cooperate against a tit-for-tat strategy whenever Eq. 16 holds.

Now suppose there is a detection mechanism that reduces the probability of reaching the state TR where Player 1 cooperates and Player 2 defects. Brams and Kilgour provide a detailed discussion about the nature of detection mechanisms and the different types of errors, but here it is assumed that the effectiveness of the mechanism is captured by the p value in

$$Pr(\text{signal Left}|\text{played Left}) = 1, Pr(\text{signal Right}|\text{played Right}) = p.$$

This simple best-case¹ detector shows how the mechanism can change the incentives for a player. Here, cooperation is preferred to defection ad infinitum if

$$(20) \quad \sum_{t=1}^{\infty} \delta^{t-1} (-|\eta_{1,2}| + \beta_{1,2}) > (1-p)(|\eta_{1,2}| + \beta_{1,2}) + p(|\eta_{2,2}| + \beta_{2,2}) \\ + \sum_{t=2}^{\infty} \delta^{t-1} (|\eta_{2,2}| + \beta_{2,2})$$

The difference between Eqs. 17 and 20 is the first term on the right hand side: In Eq. 17, the defector received the full benefit of $|\eta_{1,2}| + \beta_{1,2}$, but in Eq. 20, the benefit of defection is reduced by the $(1-p)$ probability of securing the higher benefit with the lower expected value of $(1-p)(|\eta_{1,2}| + \beta_{1,2}) + p(|\eta_{2,2}| + \beta_{2,2})$. Indeed, with no detector (so $p = 0$), Eqs. 17 and 20 are equivalent. As Eq. 20 also shows, as the detector increases in accuracy (so $p \rightarrow 1$), the expected benefit from defection, $(1-p)(|\eta_{1,2}| + \beta_{1,2}) + p(|\eta_{2,2}| + \beta_{2,2})$, decreases to $(|\eta_{2,2}| + \beta_{2,2})$.

The detection mechanism, then, makes cooperation more likely by lowering the benefit of defection. Because there are no qualitative differences comparing this analysis to the non-detection mechanism case, calculations using the decomposition provide a simpler way to analyze extensions to the standard setting, and to characterize differences between mechanisms. For instance, the detection mechanism works by the single substitution in Eq. 20, which means that other mechanisms can be contrasted by checking how they differ in terms of this modification.

3.1. Iran, Israel, and the United States. A feature developed above is how adding a third player expands the possibilities of what can happen with repeated interactions. This attribute makes it worth considering the interaction between Iran and Israel discussed in [1], but now including the United States.

In this game, Iran has a dominant strategy to develop nuclear capacity (defect), where Israel and the U.S. are seeking to sustain the cooperative outcome in which Iran does not advance its nuclear ability. As Brams and Kilgour demonstrate, a strategy detection mechanism for Israel in the two-player game induces Iran's cooperation against Iran's credible tit-for-tat strategy.

The need is to understand whether Iran might gain new options with the inclusion of the United States, given the detection mechanism possessed by Iran's opponents. What situation actually obtains depends upon the strategic structure, where Iran has a dominant strategy, but also upon the particular payoffs chosen to represent the game. Because of this, only qualitative differences are discussed to highlight problems in generalizing results that are based on simplified scenarios.

As we have shown, a characteristic of dominant strategy games with more than two players is that there can be more than a single "cooperative" outcome. (In what follows, for a two-player game, the ordering of players is (Israel, Iran); for three players it is (Israel, Iran, US).) That is, even if (C, C, C) is possible, it might also be possible to sustain (C, D, C) (where Iran defects) even if the analogous two-player situation (C, D) is unrealizable. In terms of the case study, this possibility depends on the relationship of payoffs between Israel and the U.S., which is ignored in the two-player case. As illustrated with \mathcal{G}_3 , the (C, D, C) outcome can be sustainable

¹This is a best-case detector because having $Pr(\text{signal Left}|\text{played Left}) < 1$ would decrease the incentive for cooperation by including the possibility of Player 1 defecting against Player 2's cooperation.

even if both Israel and the U.S. have strategic incentives to play D . With \mathcal{G}_3 , each player has a dominant strategy, but there are multiple sustainable states. (If Iran's strategy could, with common knowledge, be observed before the U.S. and Israel act, perhaps the likelihood of (C, D, C) could be reduced, if not eliminated all together.)

The advantage of the detection mechanism in the two player case is that the dominant strategy of defection becomes less profitable, so the advantage lies with Israel. With multiple sustainable states, however, the detection mechanism may assume another use that could benefit Iran: It could become a signaling device. Namely, Iran could signal in advance which equilibrium it prefers, perhaps (C, D, C) . If we take the commitment to develop nuclear capacity even in the face of international sanctions as a pre-commitment to defection ad infinitum, which is not an unreasonable interpretation, Iran is playing the game of equilibrium selection, acting on the belief that (C, D, C) is sustainable. An alternative possibility is where the US payoff structure tolerates a higher level of Iranian nuclear capability or places a different premium on cooperative benefits ($\beta_{i,j}$ terms) than accepted by Israel to make (C, D, C) a sustainable equilibrium in a repeated setting. Such an equilibrium would be manifested by continued Iranian nuclear activity (the D strategy), relaxed sanctions (tit-for-tat punishment is not needed for an equilibrium) even with, perhaps, Israeli objections (reflecting that (C, D) is not a two-player equilibrium). The Sect. 2.4 description suggests how to use \mathcal{G}^B to further modify the game (e.g., through negotiations) to attain other equilibria.

That this signaling feature can arise in games with multiple cooperation states was recognized and side-stepped in [1] by assuming one player had a dominant strategy. With more players, however, this strategic assumption no longer remains a valid way to avoid the problem, which means that these 2×2 results cannot be generalized beyond the two-player case. The reason for limiting the analysis to two players was to avoid the increasing number of possible games in the combinatorial approach of Robinson & Goforth [5]. Instead of this method, the foundation for analysis can be simplified to Eq. 16 or 20 by using the strategic decomposition.

4. Conclusion

Although the concerns raised here about the typical approach to analyzing repeated interactions are known, there was no efficient method of addressing them. By applying the decomposition to repeated games, analysis can be extended from isolated cases to classes of games sharing a property of interest. Furthermore, the decomposition can be used not only to analyze payoff structures, but also the different strategies and inducement mechanisms used in a repeated setting.

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DEPARTMENT OF MATHEMATICS, INSTITUTE FOR MATHEMATICAL BEHAVIORAL SCIENCES, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92697-5100

DEPARTMENT OF MATHEMATICS, INSTITUTE FOR MATHEMATICAL BEHAVIORAL SCIENCES, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92697-5100

The dynamics of consistent bankruptcy rules

Michael A. Jones and Jennifer M. Wilson

ABSTRACT. For $2 \leq k < n$, we define the k -averaging dynamic process in which solutions to k -player bankruptcy problems are used iteratively and averaged to arrive at an n -player allocation. For consistent bankruptcy rules, we prove that the n -player solution is the unique attractive fixed point of this dynamic process by adapting results from Dagan and Volij (1997). We provide alternate proofs that the fixed point is attractive for the Proportional rule (based on Markov chains) and the TAL rules (for $k = 2$). For the latter, we consider when the convergence is finite or asymptotic. For comparison, we demonstrate that an n -player solution for an inconsistent rule is not necessarily a fixed point of the dynamic process. We discuss generalizations of the dynamic process in which the averaging occurs over alternate network topologies.

1. Introduction

The Talmud is a collection of ancient texts that document and interpret Jewish criminal, civil, and religious law. The 2000-year-old Babylonian Talmud gave three examples of instances in which an estate is divided among creditors whose claims sum to more than the estate; these examples are commonly referred to as bankruptcy problems. A more general way of allocating an estate among three or more creditors was not described in the Talmud leaving Rabbinic scholars (and eventually game theorists) to wonder how the allocations in the example were determined. Of the three instances, the first case (first column in Table 1) in which an estate of 100 (of your favorite monetary unit) was divided among three claimants with claims of 100, 200, and 300 made sense because it was evenly divided among the three creditors, despite their different claim sizes. The third case (third column in Table 1) in which 300 was to be divided among the creditors was appealing because it coincided with allocating the estate proportionally based on the amount owed each creditor. The real riddle was whether a single rule was used to generate the data (as opposed to separate rules being used for different sizes of the estates) and how a single rule could describe the extremes of the first and third cases, as well as the less intuitive middle case.

The Babylonian Talmud was clearer on how to solve similar bankruptcy-type problems when there are only two claimants, an approach now known as the Contested Garment rule. The data in Table 1 has an interesting property as it relates to the Contested Garment rule. If the amount awarded to any two of the three

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Claims	Estate		
	100	200	300
100	100/3	50	50
200	100/3	75	100
300	100/3	75	150

TABLE 1. Data from the Babylonian Talmud demonstrating how estates of varying sizes should be allocated among creditors with different claims.

claimants is summed and then re-allocated to the two claimants using the Contested Garment rule, then the two claimants would receive their original amounts. For example, in the second column of Table 1, claimants 1 and 2 (with claims of 100 and 200, respectively) would receive 50 and 75, respectively, if they were to use the Contested Garment rule to divide 125 ($= 50 + 75$) based on their respective claims, matching the data from the table.

One can ask: What would the claimants do if the amounts allocated to one or more pairs of claimants did not match what they would receive under the Contested Garment rule? How could the claimants adjust the claims to arrive at an n -claimant solution? In response to these motivating questions, we introduce a dynamic process based on averaging the k -claimant solution over all sets of k players and show that for certain bankruptcy rules the process converges to the n -claimant solution. Our process generalizes a dynamic approach introduced by Dagan and Volij [5].

Others have been concerned both with how knowledge of a 2-claimant solution can be used to determine the n -claimant solution (described as a notion of consistency) and how dynamic processes can be used to converge to n -claimant solutions for bankruptcy problems and, more broadly, for cooperative games. We provide an extensive literature review in Section 2. In Section 3, we define the bankruptcy problem and review a number of well-known bankruptcy rules, including the TAL rules and the Minimal Overlap rule. We also discuss what it means for a rule to be consistent. In Section 4, we introduce the k -averaging dynamic process for bankruptcy rules and show that if the rule is consistent, then the n -claimant solution is the unique attractive fixed point of the dynamic process. We also provide an alternate proof of this result for the Proportional rule and consider an example that shows that an n -claimant solution may not be a fixed point if the rule is not consistent. In Section 5, we provide an alternate proof of the convergence of the 2-averaging dynamic process for TAL rules. At the beginning of this section, we also discuss when this convergence occurs in a finite number of iterations, and when the convergence is asymptotic; this is formalized in Proposition 5.10. Section 6 provides concluding remarks about how to extend the dynamic process to other network topologies.

2. Literature Review

The formal study of bankruptcy rules originated in the foundational papers by O'Neill [16] and Aumann and Maschler [2]. O'Neill [16] introduced the bankruptcy problem through two passages from the Babylonian Talmud. He examined the assumptions behind a solution proposed by Rabbi Ibn Ezra, and proposed an amendment in the case when no player claims the entire estate. He compared this

amended solution, now called the Minimal Overlap rule, to the results obtained using other solution methods including the method of Random Claims, the Proportional rule, and allocations based on non-cooperative game models (the Nash equilibrium) and cooperative game models (the Shapley value). He also introduced a notion of consistency which is slightly different from the current use of the term. Under this notion, for a fixed solution method, each player in turn is awarded their full claim, and the remaining estate is allocated to the remaining players using the fixed method. The results are averaged for each player, and the method is called consistent if each player's average is equal to the amount they would have received under that method if applied to the original problem.

Aumann and Maschler [2] formally defined the bankruptcy problem, and illustrated a couple of proposed solutions to bankruptcy problems in the Talmud, including the Contested Garment rule, as well as the data in Table 1. They defined a consistent solution to a bankruptcy problem as one that for each pair of players reduces to the Contested Garment rule, and showed that each bankruptcy problem has a unique consistent solution: what is now called the Talmud Rule. They showed how the Talmud rule can be viewed as a combination of two other bankruptcy rules: the Constrained Equal Awards and Constrained Equal Losses rules. For a rule to satisfy their definition of self-consistency (now called consistency), its restriction to any subset of players yields the same solution as the bankruptcy problem restricted to that subset. Finally, they modeled a bankruptcy problem as a cooperative game, and show that the Talmud rule is equal to the nucleolus of that game. This was somewhat surprising, because the nucleolus was only defined by Schmeidler [17] in 1969, yet the Talmud rule dates back thousands of years.

Others have proposed and analyzed other solutions for bankruptcy problems. Hokari and Thomson [7] proposed a class of consistent rules that associate a weight with each player to model situations in which some bias or priority in favor of some players is desired. Moreno-Tertero and Villar [15] proposed a one-parameter family of rules, called the TAL rules, that generalizes the Talmud, Constrained Equal Awards and Constrained Equal Losses rules. They also verified that the TAL rules satisfy a number of properties, including consistency, continuity and anonymity, estate and claims monotonicity, and that these rules are order-preserving and homogeneous. Moreno-Tertero [14] defined a generalization of the TAL family in the context of taxation problems (where the goal is to distribute a tax burden, and individuals' taxable incomes serve as upper bounds for their allocations), which fixes minimum and maximum tax rates. Thomson [20] defined two classes of bankruptcy rules, the ICI and CIC rules. The ICI class contains the TAL family and the Minimal Overlap rule; the CIC class contains Constrained Equal Awards, Constrained Equal Losses, as well as the Reverse Talmud rule. Thomson identified properties of each family and their consistent subclasses. Other solutions can be obtained by modeling the bankruptcy problem as a bargaining problem, a non-cooperative game or a cooperative game, and applying known solution methods. See Thomson [19] for a survey that focuses on axiomatic treatments.

The concept of consistency, which requires that a bankruptcy rule coincide with its restriction to arbitrary subsets of players, has played a fundamental role in axiomatizations and interpretations of bankruptcy problems. Although consistency has been used as a mechanism for applying the bankruptcy rule, Thomson [22] argued that consistency can best be understood as an expression of fairness by

interpreting it as a manifestation of a more general solidarity principle (in which when the population of players changes, the remaining players are affected—either positively or negatively—in the same way), in the presence of efficiency. In the context of bargaining problems, the double requirements of efficiency and solidarity imply that when a subgroup of players “leave with their portion of the estate,” the remaining players’ allocations should be unchanged. This can be construed as a type of stability.

There have been several approaches to exploring the relationship between “reduced” and initial bankruptcy problems. Hokari and Thomson [8] examined how bilateral consistency (in which a rule coincides with its restriction to subsets of size 2) can be used to infer general properties about bankruptcy rules. They defined a property to be lifted if whenever it applies to groups of 2 players, and the rule is bilaterally consistent, then the property holds for groups of players of arbitrary size. They showed that a number of well-known properties are lifted, and that several others can be lifted if resource monotonicity is assumed. Consistency has also been used to characterize allocation problems in the larger context of fair division, resource allocation and cooperative game theory. Maschler and Owen [12], for instance, defined consistency for solutions of hyperplane games by requiring each player to receive, in the solution, the average of what that player receives in all reduced hypergames corresponding to proper subsets of players to which that player belongs. Similarly, Dagan and Volij [5] defined average consistency for bankruptcy rules as follows: a rule is average consistent if each claimant’s allocation is equal to the average of his allocations in each two-player problem, and discussed how average consistency of pairs extends to n -claimant rules. Again, resource monotonicity plays a crucial role in showing that the extension of an average consistent rule is unique. Thomson [21] defined a rule as reverse consistent, if given that the rule chooses its restriction to every subgroup of the associated reduced problem, then it chooses the solution for the full group. See Thomson [21] for a survey of the role of consistency and reverse consistency in a wide context.

As suggested above, the problem of extending a general solution from 2-player solutions has been an on-going theme in bankruptcy problems, as with general allocation problems in cooperative game theory. However, the focus has largely been axiomatic rather than dynamic. For $2 \leq k < n$, our dynamic approach averages over k -player solutions to arrive at an n -player solution. Our k -averaging dynamic generalizes the pairwise averaging dynamic introduced by Dagan and Volij [5]. Their dynamic process is more of a tool to examine notions of consistency—specifically when a bilateral principle can be extended from pairs to n players. Though, they show that their 2-averaging dynamic process converges when the bankruptcy rule is monotonic in the estate size.

Relevant dynamic approaches have appeared primarily in the context of cooperative games, where several solution concepts can be considered the outcome of a sequence of bargaining or strategic moves (e.g., the nucleolus [17] can be calculated using a standard operations research algorithm). Stearns [18] defined an iterative process of transfer schemes on cooperative games. He showed that one transfer scheme converges to the kernel of the game; another converges to the game’s bargaining set. Justman [10] introduced generalized nucleoli and proposed a generalization of Stearn’s transfer scheme which defined a dynamical process that converges to the nucleolus. Yarom [23] generalized Justman’s dynamical approach

to the lexicographic kernel. These results were all based on discrete dynamical approaches; other research has focused on differential methods (see, for example, Kalai, Maschler and Owen [11]). More relevant to this article is the dynamic process outlined in Maschler and Owen [12], in which the authors define a generalized Shapley value on the set of hypergames, and then show that the solution can be obtained as the limit of an sequence of allocations. Given a Pareto optimal initial allocation, each subsequent allocation is inductively defined based on the Shapley value of a set of reduced hypergames, whose values are dependent on the previous allocation. They proved that for a wide range of initial allocations, the dynamic process converges to the correct value.

Dynamic frameworks have also been explored in other fair division problems. Moreno-Tornero [14] looked at the dynamics of voting among taxation rules, and showed that within the generalized TAL family of piecewise linear taxation rules, there is a unique taxation rule that is approved by a majority of voters. Fleurbaey and Roemer [6] defined a dynamic approach to bargaining, in which a sequence of bargaining solutions is chosen to minimize penalties derived from violating certain desired axioms, and prove that the sequence converges almost surely to the Nash bargaining solution under a wide range of penalties. Finally, Hougaard, Moreno-Tornero, and Østerdal [9] defined a mechanism to generalize bankruptcy rules to bankruptcy problems with baselines, and propose an interpretation in which a sequence of baselines is chosen in a dynamic process to capture a series of allocations over a discrete number of time periods.

3. Bankruptcy Rules and Consistency

A bankruptcy problem is described by a finite set N of n claimants (which we'll call players), a vector $\mathbf{d} \in \mathbb{R}_+^n$, where d_i is the claim of player i with $d_i \leq d_j$ for $i < j$, and an estate E , where the sum of the claims is greater than or equal to the money from the estate, i.e., $D = \sum_{i=1}^n d_i \geq E$. A bankruptcy rule R allocates the estate E among the players based on \mathbf{d} so that $R(\mathbf{d}, E)$ is an n -dimensional vector of nonnegative entries, $\sum_{i=1}^n R_i(\mathbf{d}, E) = E$, and $R_i(\mathbf{d}, E) \leq d_i$ for all i .

In this section we recall a number of well-known bankruptcy rules in preparation for the dynamic approach to be introduced in the next section. The notion of consistency is also developed, as the dynamic approach is applied and meaningful for consistent rules. The rules are compared through an example.

The Proportional rule P assigns each player its proportion $\frac{d_i}{D}$ of the estate E . For n players with $\mathbf{d} = (d_1, d_2, \dots, d_n)$ and $E \leq D$, the proportional rule assigns $P(\mathbf{d}, E) = (\frac{d_1 E}{D}, \frac{d_2 E}{D}, \dots, \frac{d_n E}{D})$.

The Constrained Equal Awards rule EA assigns to each bankruptcy problem the vector $(EA_1(\mathbf{d}, E), \dots, EA_n(\mathbf{d}, E))$ such that $EA_i(\mathbf{d}, E) = \min\{d_i, \lambda\}$, where $\lambda > 0$ is chosen so that $\sum_{i \in N} \min\{d_i, \lambda\} = E$. As the name implies, each player is awarded an equal amount up to the amount of the individual's claim.

The Constrained Equal Losses rule $EL(\mathbf{d}, E)$ assigns to each bankruptcy problem a vector $(EL_1(\mathbf{d}, E), \dots, EL_n(\mathbf{d}, E))$ such that $EL_i(\mathbf{d}, E) = \max\{0, d_i - \mu\}$, where $\mu > 0$ is chosen so that $\sum_{i \in N} \max\{0, d_i - \mu\} = E$. For small values of E , each player receives an allotment so that their losses are equal. If the estate size E is large enough, then the players with the least claim sizes will receive an amount equal to their claims, while the remaining players will have equal losses.

Aumann and Maschler [2] introduced the Talmud rule, which extends the solutions given in Table 1 to any n -player bankruptcy problem. When the estate is small relative to the sum of the claims (i.e., $E \leq D/2$), then the Talmud rule applies the Constrained Equal Awards rule to the claims vector $\mathbf{d}/2$. When the estate is large enough (i.e., $E > D/2$), then the Talmud rule applies the Constrained Equal Losses rule on $E - D/2$ for the claims vector $\mathbf{d}/2$ after allocating each player half of its claim. Introduced by Chun et al. [3], the Reverse Talmud rule does the reverse, for $E \leq D/2$: using the Constrained Equal Losses rule, and for $E > D/2$: using the Constrained Equal Awards rule on $E - D/2$ after allocating each player half of its claim.

There are a number of generalizations of the Talmud rule. One of these is the set of TAL rules introduced by Moreno-Tertero and Villar [15]. The set of TAL rules is a parametrized class of bankruptcy rules for which the Talmud rule, the Constrained Equal Awards rule and the Constrained Equal Losses rule are special cases. For a fixed $\theta \in [0, 1]$, the TAL rule with parameter θ uses the Constrained Equal Awards rule for $E \leq \theta D$ and the claims vector $\theta \mathbf{d}$ and the Constrained Equal Losses rule on $E - \theta D$ for the claims vector $(1 - \theta)\mathbf{d}$, after giving each player θ of its claim, for $E > \theta D$. (TAL derives its acronym from T for Talmud, A for awards, and L for losses.) The TAL rule with parameter θ can be written as

$$T_i^\theta(\mathbf{d}, E) = \begin{cases} EA_i(\theta \mathbf{d}, E) & \text{if } E < \theta D \\ \theta d_i + EL_i((1 - \theta)\mathbf{d}, E - \theta D) & \text{if } \theta D \leq E \leq D. \end{cases}$$

When $\theta = 1$ (resp. $\theta = 0$), the TAL rule corresponds to the Constrained Equal Awards rule (resp. the Constrained Equal Losses rule). The Talmud rule is the TAL rule for $\theta = 1/2$. As discussed previously, the 2-player Talmud rule appeared in the Talmud and is referred to as the Contested Garment rule. The following defines the 2-player TAL rule for any θ (and $d_1 \leq d_2$) by

$$T^\theta((d_1, d_2), E) = \begin{cases} (E/2, E/2) & \text{if } E < 2\theta d_1 \\ (\theta d_1, E - \theta d_1) & \text{if } 2\theta d_1 \leq E < (2\theta - 1)d_1 + d_2 \\ \left(\frac{E+d_1-d_2}{2}, \frac{E+d_2-d_1}{2}\right) & \text{if } (2\theta - 1)d_1 + d_2 \leq E \leq d_1 + d_2. \end{cases}$$

Aumann and Maschler [2] used induction to show how the Talmud rule can be calculated through a series of steps in which player 1 (owed the least amount) is awarded an amount based on the 2-player rule between player 1 and the coalition of players $\{2, \dots, n\}$, and then the player with the next least claim is awarded an amount based on the 2-player rule between player 2 and the coalition of players $\{3, \dots, n\}$, and so on. As described below, Moreno-Tertero [13] generalized this induction process for the TAL rule T^θ , calling it the θ -coalitional procedure. Assuming that the $(n - 1)$ -player T^θ is known, then the n -player solution can be divided into one of three cases.

- (1) If $E \leq n\theta d_1$, then assign equal awards to all players.
- (2) If $n\theta d_1 < E < D - n(1 - \theta)d_1$, then divide E between player 1 and $C = \{2, \dots, n\}$ using the TAL rule T^θ to solve the 2-player problem $((d_1, d_2 + \dots + d_n), E)$, using the $(n - 1)$ -player rule (which is assumed to be known by the induction hypothesis) to divide the amount allocated to coalition C between its members.
- (3) If $E \geq D - n(1 - \theta)d_1$, then assign equal losses to all players.

The three cases can be visualized on a number line representing different estate sizes $0 \leq E \leq D$ (see Figure 1). In Section 5, this coalitional approach to TAL

rules is applied to show an alternate proof of convergence under pairwise averaging dynamics. In particular, Figure 1 is expanded by breaking up the middle interval into thirds, and then repeating.

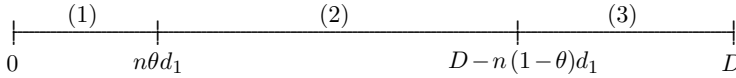


FIGURE 1. The placement of E in the tripartition of $[0, D]$ determines which case to use in the θ -coalitional approach to determining the allocation under the TAL rule T^θ .

We compare the previously defined bankruptcy rules in the following example.

EXAMPLE 3.1. We compute the allocations under the Proportional, Constrained Equal Awards, Constrained Equal Losses, and Talmud rules for the bankruptcy problem where $\mathbf{d} = (100, 200, 300, 400)$ and $E = 700$.

For the problem (\mathbf{d}, E) , $D = 100 + 200 + 300 + 400 = 1000$. Then the Proportional allocation is $P(\mathbf{d}, E) = (700/1000)\mathbf{d} = (70, 140, 210, 280)$.

For the Constrained Equal Awards rule, λ is selected to satisfy

$$\min\{100, \lambda\} + \min\{200, \lambda\} + \min\{300, \lambda\} + \min\{400, \lambda\} = 700.$$

It follows that $\lambda = 200$ and each player i is allocated $\min\{d_i, 200\}$. Hence, it follows that $EA(\mathbf{d}, E) = (100, 200, 200, 200)$.

For the Constrained Equal Losses rule, μ is selected to satisfy

$$\max\{0, 100 - \mu\} + \max\{0, 200 - \mu\} + \max\{0, 300 - \mu\} + \max\{0, 400 - \mu\} = 700.$$

It follows that $\mu = 75$, and each player i is awarded $\max\{0, d_i - \mu\}$. Hence, $EL(\mathbf{d}, E) = (25, 125, 225, 325)$.

We compute the allocation under the Talmud rule in two different ways. Because $E > 500$,

$$\begin{aligned} T^{0.5}(\mathbf{d}, E) &= (1/2)\mathbf{d} + EL((1/2)\mathbf{d}, E - (1/2)D) \\ &= (50, 100, 150, 200) + \left(0, 16\frac{2}{3}, 66\frac{2}{3}, 116\frac{2}{3}\right) = \left(50, 116\frac{2}{3}, 216\frac{2}{3}, 316\frac{2}{3}\right). \end{aligned}$$

This follows because $\mu = 83\frac{1}{3}$ satisfies $\max\{0, 50 - \mu\} + \max\{0, 100 - \mu\} + \max\{0, 150 - \mu\} + \max\{0, 200 - \mu\} = 200$ in the equal losses calculation.

Alternatively, we can use the coalitional approach to compute the Talmud rule allocation. In the first step, $4(1/2)d_1 = 200 < 700 < 800 = D - 4(1/2)d_1$ hence we apply case (2) in the coalitional algorithm, and player 1 is awarded 50 from the Contested Garment rule between player 1 and the coalition $\{2, 3, 4\}$. In step 2, the remaining amount of 650 is allocated among players 2, 3, and 4. Because the new estate size $650 \geq (D - d_1) - 3(1/2)d_2 = 600$, we apply case (3), so the remaining players' allocations is determined by $EL((200, 300, 400), 650) = (116\frac{2}{3}, 216\frac{2}{3}, 316\frac{2}{3})$.

Table 2 summarizes the allocations for the different bankruptcy rules.

All of the above rules satisfy a compelling property called consistency. Let the consistent bankruptcy rule C be used to solve a bankruptcy problem with claims vector \mathbf{d} , player set N , and estate E , so that each player i receives $C_i(\mathbf{d}, E)$.

Proportional Rule	70	140	210	280
Constrained Equal Awards	100	200	200	200
Constrained Equal Losses	25	125	225	325
Talmud Rule	50	$116\frac{2}{3}$	$216\frac{2}{3}$	$316\frac{2}{3}$

TABLE 2. Comparing four bankruptcy rules for the claims vector $\mathbf{d} = (100, 200, 300, 400)$ and $E = 700$.

Suppose that some subset of players take their allotted amounts and the remaining players (which form a subset $S \subset N$) pool their allotments, which sum to $\sum_{i \in S} C_i(\mathbf{d}, E)$. If these players re-allocate the pooled sum among the players in S using the same consistent rule C , then each player i would receive the same amount $C_i(\mathbf{d}, E)$. And, this holds for every bankruptcy problem and for all subsets S .

The following notation is needed to provide a more formal definition of consistency. For a subset $S \subseteq N$, \mathbf{d} restricted to S is $\mathbf{d}|_S$, the $|S|$ -dimensional vector $(d_{i_1}, d_{i_2}, \dots, d_{i_{|S|}})$ where $i_k \in S$ and $i_1 < i_2 < \dots < i_{|S|}$. For an n -dimensional vector \mathbf{x} , define $\mathbf{x}(S) = \sum_{i \in S} x_i$ to be the sum of the x_i 's for $i \in S$.

DEFINITION 3.1. For any bankruptcy problem (N, \mathbf{d}, E) , a bankruptcy rule C is consistent if $C(\mathbf{d}, E) = \mathbf{x}$ then, for all $S \subseteq N$ and for all $j \in N$,

$$C_j(\mathbf{d}|_S, \mathbf{x}(S)) = C_j(\mathbf{d}, E).$$

Although the previously defined rules are all consistent, not all bankruptcy rules are. The Minimal Overlap rule, originally introduced by O'Neill [16], fails to satisfy consistency. Others have worked to characterize and to more easily compute the Minimal Overlap rule. For example, Alcalde et al. [1] showed that the Minimal Overlap rule is a composition of Ibn Ezra's rule (originally considered by O'Neill [16]) and the Constrained Equal Losses rule. They also provided an axiomatic characterization of the rule. We consider an example and then give a more formal way to determine a general allocation under the Minimal Overlap rule.

EXAMPLE 3.2. [An inconsistent bankruptcy rule, part 1] Let $\mathbf{d} = (100, 200, 300)$ and $E = 300$, as in the third column of Table 1. We apply the rule presented by Ibn Ezra, as explained by O'Neill in [16]. (For this example, Ibn Ezra's method coincides with the Minimal Overlap rule.) As Ibn Ezra conceived it, each player "lays claim" to a specific portion of the estate, as visualized by the overlapping line segments on the interval E (see Figure 2). Player 3 lays claim to the entire interval or estate, while player 2 lays claim to $2/3$ of the estate and player 1 lays claim to $1/3$. As shown in the figure, the interval $[0, 100]$ is claimed by all players, so each receives $1/3$ of the interval corresponding to an equal amount of $100/3$. Only players 2 and 3 claim $[100, 200]$, so they divide it evenly, receiving $100/2$ each. Finally, player 3 receives the remaining interval $[200, 300]$, worth 100. It follows that the allocation under Ibn Ezra's method (or the Minimal Overlap rule) is $M(\mathbf{d}, E) = (100/3, 250/3, 550/3)$.

The Minimal Overlap rule specifies that each player receive an equal share of all intervals to which the player lays claim (after distributing the intervals to minimize the overlap). Each player's allocation is equal to the sum of his or her shares. The Minimal Overlap rule differs from Ibn Ezra's method when no player i has a claim $d_i \geq E$. It also specifies that if a player i has $d_i > E$, then the claim is reduced to

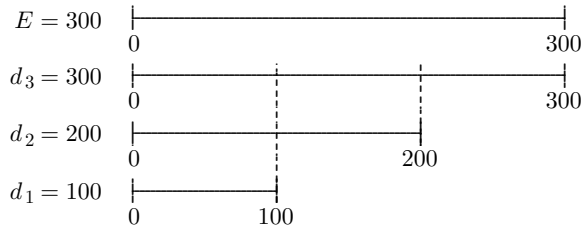


FIGURE 2. An example of the Minimal Overlap Rule for $\mathbf{d} = (100, 200, 300)$ and $E = 300$.

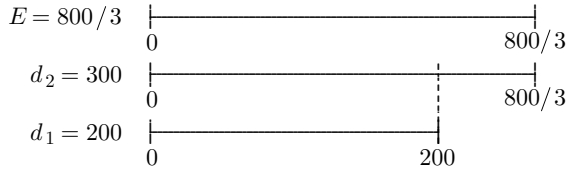


FIGURE 3. The Minimal Overlap rule is not consistent; for $\mathbf{d} = (200, 300)$ and $E = 800/3$, the Minimal Overlap rule allocation is $(300/3, 500/3)$ (instead of $(250/3, 550/3)$).

the entire estate E . These assumptions are described in the following proposition that simplifies the Minimal Overlap rule.

PROPOSITION 3.3 (Chun and Thomson [4]). *Up to relabeling parts of the amount available, there is a unique arrangement of claims achieving minimal overlap. It is obtained as follows:*

- (1) *If there is some player j such that $d_j \geq E$, then each player $i \in N$ such that $d_i \geq E$ claims $[0, E]$ and each other player k claims $[0, d_k]$.*
- (2) *If $E \geq d_i$ for each player i , then there is a unique $t \in [0, E]$ such that:*
 - (a) *each player $i \in N$ such that $d_i \geq t$ claims $[0, t]$ as well as part of $[t, E]$ of size $d_i - t$ with no overlap between claims; and*
 - (b) *each player k such that $t > d_k$ claims $[0, d_k]$.*

EXAMPLE 3.4 (An inconsistent bankruptcy rule, part 2). We use Proposition 3.3 to show that the Minimal Overlap rule is not consistent. Consider the bankruptcy problem in Example 3.2, and suppose that players 2 and 3 are to divide $800/3 = 250/3 + 550/3$ (the sum of the amounts awarded to players 2 and 3 under the Minimal Overlap rule for the problem (\mathbf{d}, E)). Player 3’s claim of 300 exceeds $800/3$ (see Figure 3). Hence player 3 claims the entire interval $[0, 800/3]$ while player 2 claims $[0, 200]$. Player 2 receives $200/2$ while player 3 receives $200/2$ plus all of the remaining estate $800/3 - 200 = 200/3$ for a total of $200/2 + 200/3 = 500/3$. Since these amounts differ from the amounts players 2 and 3 received in Example 3.2, this demonstrates that the Minimal Overlap rule is not consistent.

In the following example, we compute the Minimal Overlap allocation for the bankruptcy problem in Example 3.1. This provides an instance in which no player lays claim to the entire estate.

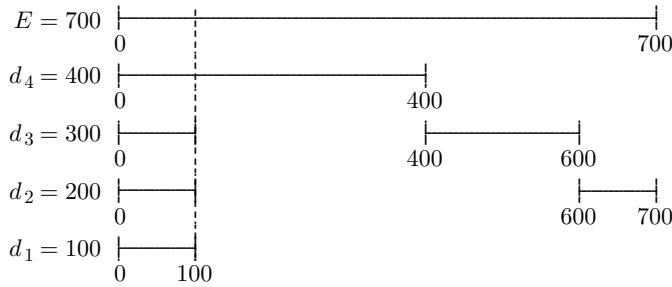


FIGURE 4. Intervals used to compute the allocation under the Minimal Overlap rule for $\mathbf{d} = (100, 200, 300, 400)$ and $E = 700$.

EXAMPLE 3.5. For $\mathbf{d} = (100, 200, 300, 400)$ and $E = 700$, the Minimal Overlap rule allocation is $(100/4, 100/4 + 100, 100/4 + 200, 100/4 + 300) = (25, 125, 225, 325)$. For this problem, $t = 100$, as t satisfies

$$t + (100 - t) + (200 - t) + (300 - t) + (400 - t) = 700.$$

The resulting minimally overlapping intervals appear in Figure 4. Each player evenly divides $[0, t]$ with players 2-4 each receiving uncontested intervals, too.

4. A Dynamic Approach to Solving the Bankruptcy Problem

For a bankruptcy problem given by an n -dimensional claims vector \mathbf{d} and E , let \mathbf{x}^0 be an initial allocation of the estate E where $\mathbf{x}^0 = (x_1^0, \dots, x_n^0)$ is any vector in the n -simplex $\{\mathbf{x} \mid x_i \geq 0 \text{ and } x_1 + \dots + x_n = E\}$. Then, \mathbf{x}^m is the allocation of the estate among the n players at round m . Let $2 \leq k < n$ and let $P_k = \{S \subset N \mid |S| = k\}$. The k -averaging dynamic process F_R^k for a bankruptcy rule R updates the allocation for player i at round $m + 1$ by averaging the allocations that player i receives when R is applied to all subsets of players of fixed size k that contain i . Specifically,

$$x_i^{m+1} = F_R^k(\mathbf{x}^m) = \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{i \in S \\ S \in P_k}} R_i(\mathbf{d}|_S, \mathbf{x}^m(S))$$

where $R_i(\mathbf{d}|_S, \mathbf{x}^m(S))$ is player i 's allocation when the bankruptcy rule R is applied to the players in S according to their initial claims and with estate size $\mathbf{x}^m(S) = \sum_{j \in S} x_j^m$.

A bankruptcy rule requires that $D = d_1 + \dots + d_n \geq E$. However, so that the k -averaging dynamic process is defined on the entire n -simplex, R must be extended for situations in which $\mathbf{x}^m(S) > \mathbf{d}(S)$. To do so, we extend R so that each player is allotted his or her claim and the excess is divided among the players in a fixed way which may depend on the original rule.

PROPOSITION 4.1. *The consistent solution $C(\mathbf{d}, E)$ for a consistent bankruptcy rule C is a fixed point of the k -averaging dynamic.*

PROOF. Let the n -player bankruptcy problem be defined by \mathbf{d} and E . The solution to the bankruptcy problem under a consistent rule C is $C(\mathbf{d}, E)$. The

k -averaging dynamic for C is defined by

$$\begin{aligned} \mathbf{x}^{m+1} &= F_C^k(\mathbf{x}^m) \\ &= \frac{1}{\binom{n-1}{k-1}} \left(\sum_{\substack{1 \in S \\ S \in P_k}} C_1(\mathbf{d}|_S, \mathbf{x}^m(S)), \dots, \sum_{\substack{n \in S \\ S \in P_k}} C_n(\mathbf{d}|_S, \mathbf{x}^m(S)) \right). \end{aligned}$$

Since C is consistent, $C_i(\mathbf{d}|_S, C(\mathbf{d}, E)(S)) = C_i(\mathbf{d}, E)$ for all players $i \in S$. Thus, substituting $\mathbf{x}^m = C(\mathbf{d}, E)$ into the equation is enough to show that the n -player allocation under C is a fixed point of the dynamic process:

$$\begin{aligned} F_C(C(\mathbf{d}, E)) &= \frac{1}{\binom{n-1}{k-1}} \left(\sum_{\substack{1 \in S \\ S \in P_k}} C_1(\mathbf{d}|_S, C(\mathbf{d}, E)), \dots, \sum_{\substack{n \in S \\ S \in P_k}} C_n(\mathbf{d}|_S, C(\mathbf{d}, E)) \right) \\ &= \frac{1}{\binom{n-1}{k-1}} \left(\sum_{\substack{1 \in S \\ S \in P_k}} C_1(\mathbf{d}, E), \dots, \sum_{\substack{n \in S \\ S \in P_k}} C_n(\mathbf{d}, E) \right) \\ &= C(\mathbf{d}, E). \end{aligned}$$

□

The following example demonstrates that the k -averaging dynamics for an inconsistent rule may not have the fixed point property.

EXAMPLE 4.2 (Example 3.2 continued). Recall that for the Minimal Overlap rule, if $\mathbf{d} = (100, 200, 300)$ and $E = 300$, then $M(\mathbf{d}, E) = (100/3, 250/3, 550/3)$. Consider $k = 2$, so that the dynamics average over all pairs. From Example 3.2, the Minimal Overlap rule is not consistent and if players 2 and 3 re-allocate $250/3 + 550/3 = 800/3$ using the Minimal Overlap rule, then the players receive 100 and $500/3$, respectively. This is one-third of the calculations for the pairwise averaging dynamics. The other two calculations involve allocating $100/3 + 250/3$ among players 1 and 2 and allocating $100/3 + 550/3$ among players 1 and 3. These allocations are 50 and $200/3$ to players 1 and 2, respectively, and 50 and $500/3$ to players 1 and 3, respectively. It follows that the allocation under the Minimal Overlap rule is not a fixed point of the pairwise averaging dynamics because

$$\begin{aligned} F_M(M(\mathbf{d}, E)) &= \frac{1}{2} [(0, 300/3, 500/3) + (150/3, 200/3, 0) + (150/3, 0, 500/3)] \\ &= (150/3, 250/3, 550/3) \neq (100/3, 250/3, 550/3) = M(\mathbf{d}, E). \end{aligned}$$

The motivation for studying the k -averaging dynamic process is to see if the n -player allocation of a consistent rule can be viewed as an outcome of an evolutionary process. The above proposition indicates that a consistent bankruptcy rule's solution is a fixed point of the dynamic process and, hence, once reached is stable. The next question is whether or not the n -player allocation is also an attractive fixed point, meaning that all initial allocations in the simplex converge to the allocation under the dynamic process defined by the consistent rule. We motivate a more general analysis by first considering the Proportional rule.

Under the k -player proportional rule, player i receives $P_i(\mathbf{d}|_S, \mathbf{x}^m(S)) = \frac{d_i}{\mathbf{d}(S)} \mathbf{x}^m(S)$ at round $m + 1$ where S is a k -player set containing i . This rule is still well-defined in the case $\mathbf{x}^m(S)$ exceeds $\mathbf{d}(S)$.

PROPOSITION 4.3. $P(\mathbf{d}, E)$ is the unique attractive fixed point for the k -averaging dynamic process F_P^k .

PROOF. The k -averaging dynamic process F_P^k can be defined by matrix multiplication so that $F_P^k(\mathbf{x}^m)$ is equal to

$$\begin{aligned} & \frac{1}{\binom{n-1}{k-1}} \left(\sum_{\substack{1 \in S \\ S \in P_k}} P_1(\mathbf{d}|_S, \mathbf{x}^m(S)), \sum_{\substack{2 \in S \\ S \in P_k}} P_2(\mathbf{d}|_S, \mathbf{x}^m(S)), \dots, \sum_{\substack{n \in S \\ S \in P_k}} P_n(\mathbf{d}|_S, \mathbf{x}^m(S)) \right) \\ &= \frac{1}{\binom{n-1}{k-1}} \begin{bmatrix} \sum_{\substack{1 \in S \\ S \in P_k}} \frac{d_1}{\mathbf{d}(S)} & \sum_{\substack{1,2 \in S \\ S \in P_k}} \frac{d_1}{\mathbf{d}(S)} & \sum_{\substack{1,3 \in S \\ S \in P_k}} \frac{d_1}{\mathbf{d}(S)} & \cdots & \sum_{\substack{1,n \in S \\ S \in P_k}} \frac{d_1}{\mathbf{d}(S)} \\ \sum_{\substack{1,2 \in S \\ S \in P_k}} \frac{d_2}{\mathbf{d}(S)} & \sum_{\substack{2 \in S \\ S \in P_k}} \frac{d_2}{\mathbf{d}(S)} & \sum_{\substack{2,3 \in S \\ S \in P_k}} \frac{d_2}{\mathbf{d}(S)} & \cdots & \sum_{\substack{2,n \in S \\ S \in P_k}} \frac{d_2}{\mathbf{d}(S)} \\ \sum_{\substack{1,3 \in S \\ S \in P_k}} \frac{d_3}{\mathbf{d}(S)} & \sum_{\substack{2,3 \in S \\ S \in P_k}} \frac{d_3}{\mathbf{d}(S)} & \sum_{\substack{3 \in S \\ S \in P_k}} \frac{d_3}{\mathbf{d}(S)} & \cdots & \sum_{\substack{3,n \in S \\ S \in P_k}} \frac{d_3}{\mathbf{d}(S)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{\substack{1,n \in S \\ S \in P_k}} \frac{d_n}{\mathbf{d}(S)} & \sum_{\substack{2,n \in S \\ S \in P_k}} \frac{d_n}{\mathbf{d}(S)} & \sum_{\substack{3,n \in S \\ S \in P_k}} \frac{d_n}{\mathbf{d}(S)} & \cdots & \sum_{\substack{n \in S \\ S \in P_k}} \frac{d_n}{\mathbf{d}(S)} \end{bmatrix} \mathbf{x}^m \\ &= \frac{1}{\binom{n-1}{k-1}} M \mathbf{x}^m. \end{aligned}$$

Because each entry of the matrix $M/\binom{n-1}{k-1}$ is positive and the columns of $M/\binom{n-1}{k-1}$ sum to 1, $M/\binom{n-1}{k-1}$ is a (right) stochastic matrix. By Perron-Frobenius theory, $M/\binom{n-1}{k-1}$ has a unique largest (in modulus) eigenvalue of 1. The associated eigenvector is the unique fixed point of the map F_P^k ; by Proposition 4.1, the eigenvector is $P(\mathbf{d}, E)$.

The eigenvector $P(\mathbf{d}, E)$ is an attractive fixed point because repeated multiplication by $M/\binom{n-1}{k-1}$ can be written in terms of its (possibly complex) eigenvalues $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$ and their associated eigenvectors $\mathbf{v}_1 = P(\mathbf{d}, E), \mathbf{v}_2, \dots, \mathbf{v}_n$ where $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n| > 0$. Because $\lim_{m \rightarrow \infty} \lambda_i^m = 0$ for $i \neq 1$, it follows that $\lim_{m \rightarrow \infty} (F_P^k)^m(\mathbf{x}^0) = F_P^k(\mathbf{d}, E)$, converging to the dominant eigenvalue/eigenvector pair, for any \mathbf{x}^0 in the n -simplex. \square

Moving from the Proportional rule, we now consider convergence for more general, consistent rules. The following proposition and theorem are adapted from Dagan and Volij [5], who consider a pairwise averaging dynamic where $k = 2$. These results indicate that the n -player solution for a consistent bankruptcy rule is the unique, attractive fixed point of the k -averaging process, as long as the rule is monotonic in the estate size. Formally, a bankruptcy rule R is monotonic in the estate size if $E \leq E'$ implies $R_i(\mathbf{d}, E) \leq R_i(\mathbf{d}, E')$ for all players i . This relatively mild condition is met by all the bankruptcy rules discussed up to this point. In particular, it is true for the Proportional rule and the set of TAL rules. Because we want the dynamic process to be defined on the entire simplex, there may be instances in which $\mathbf{x}^m(S)$ exceeds $\mathbf{d}(S)$; these are cases in which the bankruptcy

rule is not defined. The convergence results are still applicable as long as the rule is extended to these situations in such a way that the rule is still monotonic in the estate size. One way to handle this is to award $1/|S|$ of the excess to each player in S .

PROPOSITION 4.4. *If C is a monotonic bankruptcy rule, then given any two allocations, $\mathbf{x}^m \neq \mathbf{y}^m$, $|\mathbf{x}^{m+1} - \mathbf{y}^{m+1}| < |\mathbf{x}^m - \mathbf{y}^m|$ for all m , where $|\mathbf{z}|$ is defined by $|\mathbf{z}| = \sum_i |z_i|$.*

PROOF. Note that

$$\sum_i x_i^{m+1} - \sum_i y_i^{m+1} = \frac{1}{\binom{n-1}{k-1}} \sum_{\substack{i \in S \\ S \in P_k}} [C_i(\mathbf{d}|_S, \mathbf{x}^m(S)) - C_i(\mathbf{d}|_S, \mathbf{y}^m(S))].$$

It follows that

$$\begin{aligned} |\mathbf{x}^{m+1} - \mathbf{y}^{m+1}| &\leq \frac{1}{\binom{n-1}{k-1}} \sum_i \sum_{\substack{i \in S \\ S \in P_k}} |C_i(\mathbf{d}|_S, \mathbf{x}^m(S)) - C_i(\mathbf{d}|_S, \mathbf{y}^m(S))| \\ &\leq \frac{1}{\binom{n-1}{k-1}} \sum_{S \in P_k} \sum_{i \in S} |C_i(\mathbf{d}|_S, \mathbf{x}^m(S)) - C_i(\mathbf{d}|_S, \mathbf{y}^m(S))|. \end{aligned}$$

But for fixed S , the relative sizes of the estates $\mathbf{x}^m(S)$ and $\mathbf{y}^m(S)$ are fixed, so by monotonicity, the signs of $C_i(\mathbf{d}|_S, \mathbf{x}^m(S)) - C_i(\mathbf{d}|_S, \mathbf{y}^m(S))$ are the same for all $i \in S$. Thus the absolute value signs can be moved outside the sum, giving

$$\begin{aligned} |\mathbf{x}^{m+1} - \mathbf{y}^{m+1}| &\leq \frac{1}{\binom{n-1}{k-1}} \sum_{S \in P_k} \left| \sum_{i \in S} C_i(\mathbf{d}|_{S \setminus i}, \mathbf{x}^m(S \setminus i)) - C_i(\mathbf{d}|_{S \setminus i}, \mathbf{y}^m(S \setminus i)) \right| \\ &= \frac{1}{\binom{n-1}{k-1}} \sum_{S \in P_k} |\mathbf{x}^m(S) - \mathbf{y}^m(S)| \\ &\leq \frac{1}{\binom{n-1}{k-1}} \sum_{S \in P_k} \sum_{i \in S} |x_i^m - y_i^m| = \sum_i |x_i^m - y_i^m| \end{aligned}$$

since for each i , there are $\binom{n-1}{k-1}$ coalitions $S \in P_k$ that contain i . Moreover, we claim this last inequality is strict. To see this note that since $\mathbf{x}^m \neq \mathbf{y}^m$, there exists players i, j such that $x_i^m < y_i^m$ and $x_j^m > y_j^m$. Hence, if $i, j \in S \in P_k$, then

$$\begin{aligned} |\mathbf{x}^m(S) - \mathbf{y}^m(S)| &= |(x_i^m - y_i^m) + (x_j^m - y_j^m) + [\mathbf{x}^m(S \setminus i, j) - \mathbf{y}^m(S \setminus i, j)]| \\ &< |(x_i^m - y_i^m)| + |(x_j^m - y_j^m)| + |\mathbf{x}^m(S \setminus i, j) - \mathbf{y}^m(S \setminus i, j)| \\ &\leq \sum_{i \in S} |x_i^m - y_i^m|. \end{aligned}$$

This proves the claim and the Proposition. □

A consequence of Proposition 4.4 is that for any given bankruptcy problem, F_C^k has a unique fixed point, $C(\mathbf{d}, E)$. Further, the fixed point is attractive, as proved below.

THEOREM 4.5. *$C(\mathbf{d}, E)$ is the unique attractive fixed point for the k -averaging dynamic process F_C^k .*

PROOF. By Proposition 4.4, $|\mathbf{x}^m - C(\mathbf{d}, E)|$ is a monotonically decreasing sequence and hence converges to some $\epsilon \geq 0$. In addition, since $\{\mathbf{x}^m\}_{m \geq 0}$ is an infinite sequence in the compact set $\{\mathbf{x} \mid x_i \geq 0 \text{ and } x_1 + \dots + x_n = E\}$, it has a convergent subsequence $\mathbf{x}^{m_k} \rightarrow \mathbf{x}^*$ for some allocation \mathbf{x}^* . Hence $|\mathbf{x}^{m_k} - C(\mathbf{d}, E)| \rightarrow |\mathbf{x}^* - C(\mathbf{d}, E)|$; since \mathbf{x}^{m_k} is a subsequence of \mathbf{x}^m , this implies $|\mathbf{x}^* - C(\mathbf{d}, E)| = \epsilon$. Likewise, since C is continuous, $|F_C(\mathbf{x}^{m_k}) - C(\mathbf{d}, E)| \rightarrow |F_C(\mathbf{x}^*) - C(\mathbf{d}, E)|$; since $F_C(\mathbf{x}^{m_k})$ is a subsequence of \mathbf{x}^m , we also have $|F_C(\mathbf{x}^*) - C(\mathbf{d}, E)| = \epsilon$. So $|F_C(\mathbf{x}^*) - C(\mathbf{d}, E)| = |\mathbf{x}^* - C(\mathbf{d}, E)|$. Proposition 4.4 then implies $\mathbf{x}^* = C(\mathbf{d}, E)$. The fact that $\mathbf{x}^m \rightarrow \mathbf{x}^*$ follows, since $|\mathbf{x}^m - \mathbf{x}^*| = |\mathbf{x}^m - C(\mathbf{d}, E)|$ monotonically decreases and $|\mathbf{x}^{m_k} - \mathbf{x}^*| \rightarrow 0$. \square

5. A Closer Look at Pairwise Averaging Dynamics for TAL Rules

In this section, we look more closely at the dynamics of the TAL rules when $k = 2$. To gain some insight, we consider two examples that demonstrate different properties of convergence. In the first example, the convergence is in finite time, while in the second example, the convergence is asymptotic. The Talmud rule is used in both examples.

Although we defined the 2-player TAL rule T^θ previously, we restate the definition below in the iterative context, with a more compact notation. For $d_i \leq d_j$, let

$$T(i, j, m) = \begin{cases} \frac{x_i^m + x_j^m}{2} & \text{if } x_i^m + x_j^m < 2\theta d_i \\ \theta d_i & \text{if } 2\theta d_i \leq x_i^m + x_j^m < (2\theta - 1)d_i + d_j \\ \frac{x_i^m + x_j^m + d_i - d_j}{2} & \text{if } (2\theta - 1)d_i + d_j \leq x_i^m + x_j^m \leq d_i + d_j \end{cases}$$

(we suppress the θ when it is clear from context).

EXAMPLE 5.1 (Convergence in finite time). Let $\mathbf{d} = (100, 200, 300)$ and $E = 200$. The Talmud rule allocation is $T^{0.5}(\mathbf{d}, E) = (50, 75, 75)$, which is the second column in Table 1. Let $\mathbf{x}^0 = (30, 85, 85)$. The first iteration under the pairwise averaging dynamics F_T can be calculated by the Contested Garment rule, where $T(1, 2, 1) = 50$, $T(1, 3, 1) = 50$, and $T(2, 3, 1) = 85$, so that

$$F_T(\mathbf{x}^0) = \frac{1}{2} [(50, 65, 0) + (50, 0, 65) + (0, 85, 85)] = (50, 75, 75).$$

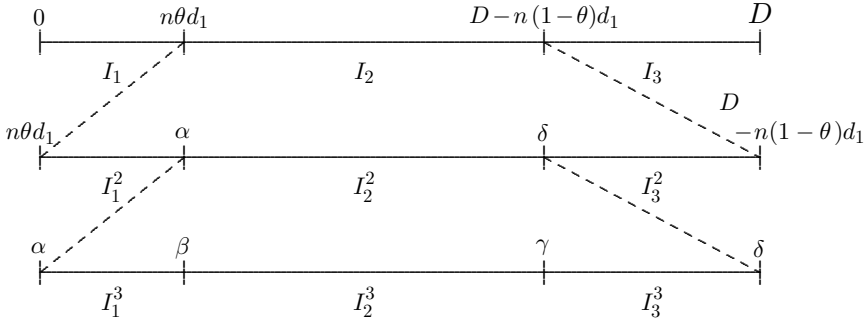
Because $F_T(\mathbf{x}^0) = (50, 75, 75) = T^{0.5}(\mathbf{d}, E)$, the pairwise averaging dynamics converges to the Talmud solution in a single step.

EXAMPLE 5.2 (Convergence asymptotically). Let $\mathbf{d} = (100, 200, 300)$ and $E = 300$. The Talmud rule allocation is $T^{0.5}(\mathbf{d}, E) = (50, 100, 150)$, which is the third column in Table 1. Suppose that $\mathbf{x}^0 = (100, 100, 100)$. The first iteration under the pairwise averaging dynamics F_T can be calculated by the Contested Garment rule, where $T(1, 2, 1) = 50$, $T(1, 3, 1) = 50$, and $T(2, 3, 1) = 100$ so that

$$F_T(\mathbf{x}^0) = \frac{1}{2} [(50, 150, 0) + (50, 0, 150) + (0, 100, 100)] = (50, 125, 125).$$

Subsequent iterations yield $F_T^m(\mathbf{x}^0) = (50, 100 + 50(1/2)^m, 150 - 50(1/2)^m)$ so that the pairwise averaging dynamics converges asymptotically to the Talmud solution because $\lim_{m \rightarrow \infty} F_T^m(\mathbf{x}^0) = (50, 100, 150) = T(\mathbf{d}, E)$.

The θ -coalitional procedure of Moreno-Tornero [13] can help determine when the pairwise dynamic for the TAL rule converges in finite time and when it is asymptotic. Let the claim sizes be divided into r distinct values $d_{i_1} < \dots < d_{i_r}$, and let $A_k = \{l \mid d_l = d_{i_k}\}$ with $a_k = |A_k|$ for $k = 1, \dots, r$. Consider Figure 5, in which the E number line is broken into a series of intervals. Each picture replicates the tripartition of Figure 1 by subdividing the middle interval of the previous picture. (We assume that the middle interval is open in each case.)



where
 $\alpha = a_1\theta d_1 + (n - a_1)\theta d_{i_2}$
 $\beta = a_1\theta d_1 + a_2\theta d_{i_2} + (n - a_1 - a_2)\theta d_{i_3}$
 $\gamma = D - [a_1(1 - \theta)d_1 + a_2(1 - \theta)d_{i_2} + (n - a_1 - a_2)(1 - \theta)d_{i_3}]$
 $\delta = D - [a_1(1 - \theta)d_1 + (n - a_1)(1 - \theta)d_{i_2}]$

FIGURE 5. The E number line is partitioned by iterating the tripartition from the θ -coalitional approach. Stage 1 divides the number line into 3 intervals: I_1, I_2, I_3 ; Stage 2 subdivides I_2 into I_1^2, I_2^2, I_3^2 ; Stage 3 subdivides I_2^2 into I_1^3, I_2^3, I_3^3 .

The relationship between Figure 5 and the θ -coalitional procedure is most easily described when the claims are distinct and $A_i = i$ for $i = 1, \dots, n$. Using the numbering system from the earlier discussion of this procedure, the top-most picture corresponds to Step 1 in which E is allocated between player 1 and the coalition $\{2, \dots, n\}$; the intervals I_1, I_2 and I_3 correspond to Cases (1), (2) and (3) respectively. In I_1 players share equal awards; in I_3 players share equal losses; in I_2 player 1 receives θd_1 and the coalition $\{2, \dots, n\}$ receives the remainder. The second picture corresponds to Step 2 in which the total allocation for players $\{2, \dots, n\}$ in Step 1 is reallocated between players 2 and the coalition $\{3, \dots, n\}$; again, the intervals I_1^2, I_2^2 and I_3^2 correspond to Cases (1), (2) and (3). In I_1^2 all players in the coalition share equal awards; in I_3^2 all players in the coalition share equal losses; and in I_2^2 player 2 receives θd_2 and the coalition $\{3, \dots, n\}$ receives the remainder. The third picture corresponds to Step 3, and so on. If the claims are not distinct then the top-most picture applies to the first a_1 steps, in which the allocation of all players $l \in A_1$ is determined. In I_1 all players share equal awards; in I_3 all players share equal losses; in I_2 , all players $l \in A_1$ receive θd_1 and the coalition of all players $p \notin A_1$ receives the remainder.

Returning to the question of finite versus asymptotic convergence, suppose the claims are distinct. We claim that if $E \in I_2$, then the allocations x_1^m converge

in finite time; if $E \in I_1 \cup I_3$, the convergence might be asymptotic. Similarly, if $E \in I_2^2$, the allocations x_2^m converge in finite time; if $E \in I_1^2 \cup I_3^2$, the convergence might be asymptotic. In general, if $a_k = 1$ so that $A_k = \{p\}$ for some player p with unique claim, then x_p^m converges in finite time if $E \in I_2^k$; if $E \in I_1^k \cup I_3^k$ the convergence might be asymptotic.

These claims are illustrated in the following theorem, whose purpose is to present an alternate, direct proof of the convergence of the TAL rules when $k = 2$. The proof is based on the intervals in which E lies and mirrors the structure of the θ -coalitional procedure.

THEOREM 5.3. *Let $T(\mathbf{d}, E)$ be the TAL rule solution to a given bankruptcy problem, and let (x_1^0, \dots, x_n^0) be an initial allocation among the players such that $\sum_{i \leq n} x_i^0 = E$. For each player i , let $x_i^{m+1} = \frac{1}{n-1}[\sum_{j \neq i} T(i, j, m)]$. Then the sequence x_i^m converges to $T_i(\mathbf{d}, E)$.*

Before beginning the proof of the theorem, we note the following. Suppose that $d_i \leq d_j$ and $x_i^M \leq x_j^M$ for some M . Then it is easy to show that $x_i^{M+1} \leq x_j^{M+1}$, and hence $x_i^m \leq x_j^m$ for all $m \geq M$. So as the iteration proceeds, if the ordering of the players' allocation change, it only changes in such a way as to decrease the "lexicographic" order of the indices. (For instance if $x_1^m \leq x_2^m \leq x_4^m \leq x_3^m$ then in the next iteration, either the ordering stays the same or $x_1^{m+1} \leq x_2^{m+1} \leq x_3^{m+1} \leq x_4^{m+1}$.) Similar remarks apply to the $d_i - x_i^m$'s.

Thus, whatever the order of the initial allocation, there exists an M such that the order of both the x_i^m 's (the "wins") and the $d_i - x_i^m$'s (the "losses") is fixed for all $m \geq M$. Denote this order by $x_{i_1} \leq x_{i_2} \leq \dots \leq x_{i_n}$ and $d_{j_1} - x_{j_1}^m \leq d_{j_2} - x_{j_2}^m \leq \dots \leq d_{j_n} - x_{j_n}^m$ for all $m \geq M$ where the i_k and are chosen inductively in the following manner: $i_1 = \min\{l \mid x_l^m \leq x_p^m \text{ for all } p\}$ and $i_k = \min\{l \neq i_{k-1} \mid x_{i_{k-1}}^m \leq x_l^m \leq x_p^m \text{ for all } p \neq i_{k-1}\}$ (and similarly for the j_k 's).

The proof of the Theorem is organized around a sequence of Stages corresponding to the steps in the θ -coalitional procedure.

STAGE 1: We consider the three intervals I_1, I_2 and I_3 and show

- If $E \in I_1$ then all players share equal awards (less than or equal to θd_1).
- If $E \in I_3$ all players share equal losses (greater than or equal to $d_i - \theta d_1$).
- If $E \in I_2$, then x_l^m converges to θd_1 for all players $l \in A_1$. If $a_1 = 1$ then this convergence occurs in finite time.

This concludes the proof of convergence for all players $l \in A_1$ and for all players $p \notin A_1$ for $E \in I_1 \cup I_3$.

STAGE 2: We consider the three sub-intervals of I_2, I_1^2, I_2^2 and I_3^2 and show

- If $E \in I_1^2$ then all remaining players share awards (at least θd_1 and less than or equal to θd_{i_2}).
- If $E \in I_3^2$ all remaining players share losses (greater than or equal to $d_i - (1 - \theta)d_{i_2}$ but at most $d_i - (1 - \theta)d_1$).
- If $E \in I_2^2$, then x_l^m converges to θd_{i_2} for all players $l \in A_2$. If $a_2 = 1$ then this convergence occurs in finite time.

This concludes the proof of convergence for all players $l \in A_1 \cup A_2$ and for all players $p \notin (A_1 \cup A_2)$ for $E \in I_1^2 \cup I_3^2$.

The Stages continue until the proof is complete.

Before beginning, we require a couple of preliminary lemmas. The proofs of these lemmas appear in the Appendix.

LEMMA 5.4. (a) If $x_i^m \leq x_k^m$ and $x_i^m \leq d_i$ then $T(i, k, m) \geq x_i^m$. (b) If $d_j - x_j^m \leq d_k - x_k^m$ and $x_j^m \geq \theta d_j$ then $T(j, k, m) \leq x_j^m$.

LEMMA 5.5. (a) Suppose that $x_q^M \leq \theta d_1 - \delta$ for some q , M and $\delta > 0$. If $x_p^M \geq \theta d_1$ for some player p , then $T(p, q, M) \leq x_p^M - \delta/2$. (b) If $x_q^M = \theta d_1$ and $x_p^M \leq \theta d_1 - \lambda$ for some $\lambda > 0$, then $T(p, q, M) \geq x_p^M + \lambda/2$. (c) If $x_q^M \geq \theta d_1 + \delta$ for some $\delta > 0$ and $x_p^M \leq \theta d_1 + \lambda$ for some $\lambda \leq \beta = \min\{\delta/2, \theta(d_{i_2} - d_1)\}$ and player $p \notin A_1$, then $T(p, q, M) \geq x_p^M + \beta/2$.

Moving to the proof of the Theorem, we note that the intent of the following propositions is similar: to show that, except when E lies on a boundary of one of the intervals listed in Figure 5, after a finite number of iterations, the rule that is applied to determine the value of $T(i, j, m)$ for each pair of players i and j remains constant. That is, there exists an M such that for each i, j with $d_i \leq d_j$, either $T(i, j, m) = (x_i^m + x_j^m)/2$ for all $m \geq M$, $T(i, j, m) = x_i^m + x_j^m - \theta d_i$ for all $m \geq M$ or $T(i, j, m) = (x_i^m + x_j^m + d_i - d_j)/2$ for all $m \geq M$.

STAGE 1

PROPOSITION 5.6. If $E < n\theta d_1$ then x_i^m converges to E/n for all players i .

PROOF. Suppose that $E = n(\theta d_1 - \delta)$ for some $\delta > 0$, so that $x_{i_1}^m \leq \theta d_1 - \delta$ for all m .

Step 1: We claim there exists an M_1 such that $d_p - x_p^m \geq (1 - \theta)d_1$ for all players p and $m \geq M_1$. Suppose that $d_{j_1} - x_{j_1}^M < (1 - \theta)d_1$ for some M , and let $\lambda > 0$ be such that $d_{j_1} - x_{j_1}^M = (1 - \theta)d_1 - \lambda$ or $x_{j_1}^M = d_{j_1} - (1 - \theta)d_1 + \lambda$. By Lemmas 5.4 and 5.5, $T(j_1, i_1, M) \leq x_{j_1}^M - \delta/2$ and $T(j_1, p, M) \leq x_{j_1}^M$ for all $p \neq j_1$ respectively. Hence

$$\begin{aligned} x_{j_1}^{M+1} &\leq \frac{1}{n-1} \left[\sum_{p \neq j_1, i_1} x_{j_1}^M + (x_{j_1}^M - \delta/2) \right] = x_{j_1}^M - \delta/2(n-1) \\ &= d_{j_1} - (1 - \theta)d_1 + \lambda - \delta/2(n-1). \end{aligned}$$

or $d_{j_1} - x_{j_1}^{M+1} \geq (1 - \theta)d_1 - \lambda + \delta/2(n-1)$. Since the quantity $\delta/2(n-1)$ is fixed, we see that by repeating the process, we can find an M_1 such that $d_{j_1} - x_{j_1}^{M_1} \geq (1 - \theta)d_1$, and hence $d_p - x_p^M \geq (1 - \theta)d_1$ for all players p . It is easy to show that this remains the case for all $m \geq M_1$.

Remark 1: The claim implies $x_l^m \leq \theta d_1$ for all $l \in A_1$ and $m \geq M_1$, and that $x_p^m + x_l^m \leq d_p + (2\theta - 1)d_1$ for all players p . So if $l \in A_1$ and $p \notin A_1$ then either $T(p, l, m) = (x_l^m + x_p^m)/2 \leq \theta d_1$ or $T(p, l, m) = x_l^m + x_p^m - \theta d_1 \leq x_p^m$; if $x_p^m \geq \theta d_1$, then $T(p, l, m) \leq x_p^m$.

Step 2: We claim that there exists an $M_2 \geq M_1$ such that $x_p^m \leq d_p - (1 - \theta)d_{i_2}$ for all $p \notin A_1$ and $m \geq M_2$. Suppose $d_j - x_j^M < (1 - \theta)d_{i_2}$, where $j \notin A_1$ is the player not in A_1 with the smallest loss, and let $\lambda > 0$ be such that $d_j - x_j^M = (1 - \theta)d_{i_2} - \lambda$ or $x_j^M = d_j - (1 - \theta)d_{i_2} + \lambda$. Again, by Lemmas 5.4 and 5.5, $T(j, i_1, M_1) \leq x_j^M - \delta/2$, $T(j, p, M_1) \leq x_j^M$ for all $p \notin A_1$, and as noted in Remark 1, $T(j, l, M_1) \leq x_j^M$ for all $l \in A_1$. The proof continues as in Step 1.

Remark 2: The claim implies $x_l^m \leq \theta d_{i_2}$ for all $l \in A_2$ and $m \geq M_2$, and that $x_p^m + x_l^m \leq d_p + (2\theta - 1)d_{i_2}$ for all players $p \notin A_1$. So if $l \in A_2$ and $p \notin (A_1 \cup A_2)$,

then either $T(p, l, m) = (x_l^m + x_p^m)/2 \leq \theta d_{i_2}$ or $T(p, l, m) = x_l^m + x_p^m - \theta d_{i_2} \leq x_p^m$; if $x_p^m \geq \theta d_{i_2}$, then $T(p, l, m) \leq x_p^m$.

Step 3: By repeating the argument in Steps 1 and 2, we can show that for each $k \leq r$ there exists an $M_k \geq M_{k-1}$ such that $x_p^m \leq d_p - (1 - \theta)d_{i_k}$ for all players $p \notin (A_1 \cup \dots \cup A_{k-1})$ and $m \geq M_k$. (This implies $x_l^m \leq \theta d_{i_k}$ for all $l \in A_k$ and $m \geq M_k$.) Thus, we obtain finally, an M_{r-1} such that $x_p^m \leq d_p - (1 - \theta)d_{r-1}$ for all $p \in A_r$ and $x_l^m \leq \theta d_l$ for all $l \in A_k$, $k \leq r - 1$, and for all $m \geq M_{r-1}$.

Step 4: We claim there exists an $M' \geq M_{r-1}$ such that $x_p^m \leq \theta d_1$ for all players p and $m \geq M'$. Suppose that $x_{i_n}^M > \theta d_1$ for some $M \geq M_{r-1}$, and let $x_{i_n}^M = \theta d_1 + \lambda$. By Lemma 5.5, $T(i_n, i_1, M) \leq x_{i_n}^M - \delta/2$. Now consider the value of $T(i_n, p, M)$ for some player $p \neq i_n$. If $d_{i_n} \leq d_p$ then $T(i_n, p, M) \leq (x_{i_n}^M + x_p^M)/2 \leq x_{i_n}^M$. If $d_{i_n} > d_p$ then $x_{i_n}^M + x_p^M \leq d_{i_n} + (2\theta - 1)d_p$, and so either $T(i_n, p, M) = (x_{i_n}^M + x_p^M)/2 \leq x_{i_n}^M$ or $T(i_n, p, M) = x_{i_n}^M + x_p^M - \theta d_p \leq x_{i_n}^M$. Thus

$$x_{i_n}^{M+1} \leq \frac{1}{n-1}[(n-2)x_{i_n}^M + (x_{i_n}^M - \delta/2)] \leq x_{i_n}^M - \delta/2(n-1) = \theta d_1 + \lambda - \delta/2(n-1).$$

Since $\delta/2(n-1)$ is a fixed quantity, we see that by repeating the process, we can find an M' that satisfies the claim.

Step 5: By Step 4, $x_p^m \leq \theta d_1$ and hence $T(p, q, m) = (x_p^m + x_q^m)/2$ for all players p, q and $m \geq M'$. Thus

$$x_p^{m+1} = \frac{1}{n-1} \left[\sum_{q \neq p} \frac{x_p^m + x_q^m}{2} \right] = \frac{1}{2} x_p^m + \frac{1}{2(n-1)} \sum_{q \neq p} x_q^m.$$

Converting these n equations into matrix format we see that the coefficient matrix forms a stochastic matrix, like the one in Proposition 4.3 for the Proportional rule. It has a unique attractive fixed point at $x_p = E/n$ for all players p . \square

PROPOSITION 5.7. *Suppose that $E > \sum_p [d_p - (1 - \theta)d_1]$. Then x_p^m converges to $d_p - (1 - \theta)d_1 + [E - \sum_p (d_p - (1 - \theta)d_1)]/n$ for all players p .*

PROOF. The proof is analogous to that of Proposition 5.6. \square

PROPOSITION 5.8. *Suppose that $n\theta d_1 \leq E \leq \sum_p [d_p - (1 - \theta)d_1]$. Then for every $\epsilon > 0$ there exists an M such that $\theta d_1 - \epsilon \leq x_p^m \leq d_p - (1 - \theta)d_1 + \epsilon$ for all $m \geq M$ and players p .*

PROOF. Note that $E \geq n\theta d_1$ implies $x_{i_n}^m \geq \theta d_1$ for all m .

Step 1: We claim for every $\epsilon > 0$ there exists an M_1 such that $x_p^m \geq \theta d_1 - \epsilon$ for all players p and $m \geq M_1$. Suppose that $x_{i_1}^M < \theta d_1$ for some M and let $x_{i_1}^M = \theta d_1 - \lambda$ for some $\lambda > 0$. By Lemmas 5.4 and 5.5, $T(i_1, i_n, M) \geq x_{i_1}^M + \lambda/2$ and $T(i_1, p, M) \geq x_{i_1}^M$ for all players $p \neq i_1$ or i_n . Hence

$$x_{i_1}^{M+1} \geq \frac{1}{n-1}[(n-1)x_{i_1}^M + \lambda/2] = x_{i_1}^M + \frac{1}{2(n-1)}\lambda = \theta d_1 - [1 - \frac{1}{2(n-1)}]\lambda.$$

By repeating the argument, we can show that $x_{i_1}^{M+2} \geq \theta d_1 - c^2\lambda$ and in general that $x_{i_1}^{M+t} \geq \theta d_1 - c^t\lambda$ where $c = 1 - \frac{1}{2(n-1)} < 1$. This proves the claim.

Step 2: A similar argument can be made to show that for every $\epsilon > 0$ there exists an $M_2 \geq M_1$ such that $x_p^m \leq d_p - (1 - \theta)d_1 + \epsilon$ for all players p and $m \geq M_2$. \square

Applying Proposition 5.8 to $l \in A_1$, we have the following.

COROLLARY 5.9. *If $n\theta d_1 \leq E \leq \sum_p [d_p - (1 - \theta)d_1]$, then x_l^m converges to θd_1 for all $l \in A_1$.*

PROPOSITION 5.10. *(a) If $E = n\theta d_1$ then x_p^m converges to θd_1 for all players p . (b) If $E = \sum_p [d_p - (1 - \theta)d_1]$ then x_p^m converges to $d_p - (1 - \theta)d_1$. (c) If $n\theta d_1 < E < \sum_p [d_p - (1 - \theta)d_1]$ then there exists an M such that $2\theta d_1 \leq x_l^m + x_p^m \leq d_p + (2\theta - 1)d_1$ for all $l \in A_1$, $p \notin A_1$ and $m \geq M$. Thus, if $A_1 = \{1\}$, $x_1^{m+1} = \frac{1}{n-1} \sum_{p \neq 1} T(1, p, m) = \frac{1}{n-1} \sum_{p \neq 1} \theta d_1 = \theta d_1$ for all $m \geq M$, so x_1^m converges to the correct value in a finite number of iterations.*

PROOF. (a). Suppose that $E = n\theta d_1$. By Proposition 5.8, given $\epsilon > 0$ there is an M such that $\theta d_1 - \epsilon < x_p^m$ for all players p and $m \geq M$. So $x_{i_n}^m = E - \sum_{i \neq i_n} x_i^m \leq n\theta d_1 - (n - 1)(\theta d_1 - \epsilon) = \theta d_1 + (n - 1)\epsilon$. Thus $\theta d_1 - \epsilon < x_p^m \leq x_{i_n}^m \leq \theta d_1 + (n - 1)\epsilon$ for all players p . Since ϵ was arbitrary, x_p^m converges to θd_1 for every player p .

(b) The proof is analogous to (a).

(c) Let $\delta > 0$ be such that $E = n(\theta d_1 + \delta)$, so $x_{i_n}^m \geq \theta d_1 + \delta$ for every m . We claim that there exists an M such that $x_l^m + x_p^m \geq 2\theta d_1$ for all $l \in A_1$, $p \notin A_1$ and $m \geq M$. Note that this is trivial if $\theta = 0$, so assume $\theta > 0$. By the Corollary, we can pick M such that $\theta d_1 - \epsilon < x_l^m < \theta d_1 + \epsilon$ for all $l \in A_1$ and $m \geq M$, where $\epsilon > 0$ is chosen below. It is easy to show that this implies $x_p^m - 2\epsilon \leq T(p, l, m) < x_p^m + 2\epsilon$ for all players $l \in A_1$, $p \notin A_1$ and $m \geq M$.

Now let $i' \notin A_1$ be such that $x_{i'}^m \leq x_{i_n}^m$ for all $i \notin A_1$. Note that if $x_{i'}^M \geq \theta d_1 + \epsilon$ then $x_p^M + x_{i'}^M \geq x_{i'}^M + x_l^M \geq 2\theta d_1$ for all $l \in A_1$ and $p \notin A_1$. So assume $x_{i'}^M = \theta d_1 + \lambda$ where $-\epsilon \leq \lambda < \epsilon$. If $\epsilon \leq \beta = \min\{\delta/2, \theta(d_{i_2} - d_1)\}$, then by Lemmas 5.5 (b) and 5.4, $T(i', i_n, M) \geq x_{i'}^M + \beta/2$ and $T(i', p, M) \geq x_{i'}^M$ for all players $p \notin A_1$. Hence

$$\begin{aligned} x_{i'}^{M+1} &\geq x_{i'}^M + \frac{1}{n-1}[-2n\epsilon + \beta/2] = \theta d_1 + \lambda + \frac{1}{n-1}[-2n\epsilon + \beta/2] \\ &\geq \theta d_1 - \epsilon + \frac{1}{n-1}[-2n\epsilon + \beta/2] = \theta d_1 + \epsilon + \frac{1}{n-1}[(-4n + 2)\epsilon + \beta/2]. \end{aligned}$$

So $x_{i'}^{M+1} \geq \theta d_1 + \epsilon$ if $\epsilon \leq \beta/2(4n - 2)$. It is easy to show that this remains the case for all $m \geq M + 1$, and hence $x_l^{M+1} + x_p^{M+1} \geq x_l^{M+1} + x_{i'}^{M+1} \geq 2\theta d_1$ for all $l \in A_1$, $p \notin A_1$ and $m \geq M + 1$. The remaining half of the inequality is shown similarly. \square

This concludes Stage 1. Note that we proved the Theorem for $E \in I_1 \cup I_3$. For $E \in I_2$, we have shown x_l^m converges to the correct value for all $l \in A_1$ and that the pairwise allocations are given by $T(p, l, m) = x_l^m + x_p^m - \theta d_1$ for each $l \in A_1$, and $p \notin A_1$, for sufficiently large m .

STAGE 2

PROPOSITION 5.11. *Suppose $n\theta d_1 < E < \sum_p [d_p - (1 - \theta)d_1]$ and $a_1 = |A_1|$. (a) If $n\theta d_1 < E < a_1\theta d_1 + (n - a_1)\theta d_{i_2}$ then x_p^m converges to $\theta d_1 + (E - n\theta d_1)/(n - a_1)$ for all players $p \notin A_1$. (b) If $a_1\theta d_1 + \sum_{p \notin A_1} [d_p - (1 - \theta)d_{i_2}] < E < \sum_p [d_p - (1 - \theta)d_1]$ then x_p^m converges to $d_p - \delta$ where $\delta = [\sum_{p \notin A_1} d_p - (E - a_1\theta d_1)]/(n - a_1)$.*

PROOF. We outline the proof of (a). Let $E = a_1\theta d_1 + (n - a_1)(\theta d_{i_2} - \delta)$ for some $\delta > 0$. By Proposition 5.10, we can pick M such that $\sum_{l \in A_1} |x_l^m - \theta d_1| < \delta$

for all $m \geq M$ and $R(p, l, m) = x_p^m + x_l^m - \theta d_1$ for all players $p \notin A_1$, $l \in A_1$ and $m \geq M$. Note that this implies that $x_{i_1}^m \leq \theta d_{i_2} - \delta$ for all $m \geq M$ where i_1 is the player with the smallest allocation among all players not in A_1 . The proof proceeds as in Proposition 5.6.

At its conclusion, we find that there exists an M_1 such that $x_p^m + x_q^m < 2\theta d_{i_2}$ for all players $p, q \notin A_1$ and $m \geq M_1$. Thus if $q \notin A_1$,

$$\begin{aligned} x_q^{m+1} &= \frac{1}{n-1} \left[\sum_{l \in A_1} (x_q^m + x_l^m - \theta d_1) + \sum_{p \notin A_1, p \neq q} (x_q^m + x_p^m) / 2 \right] \\ &= \frac{1}{n-1} \left[\sum_{l \in A_1} (x_l^m - \theta d_1) + \frac{(n+a_1-1)}{2} x_q^m + \frac{1}{2} \sum_{p \notin A_1, p \neq q} x_p^m \right]. \end{aligned}$$

To show that this system of equations converges, we use a change of variables. Let $y_1^m, \dots, y_{n-a_1}^m$ be the set of allocations $x_{i_1}^{m+M_1}, \dots, x_{i_{n-a_1}}^{m+M_1}$ of all players $l_s \notin A_1$ for $m \geq 0$. Then

$$(5.1) \quad \mathbf{y}^{m+1} = M\mathbf{y}^m + \mathbf{b}_m$$

where M is an $(n-a_1)$ by $(n-a_1)$ matrix with diagonal entries equal to $\frac{(n+a_1-1)}{2(n-1)}$ and non-diagonal entries equal to $\frac{1}{2}$, and \mathbf{b}_m is an $n-a_1$ dimensional vector with all coordinates equal to $\frac{1}{n-1} \sum_{l \in A_1} (x_l^{m+M_1} - \theta d_1)$. Note that $\mathbf{b}_0 = \frac{1}{n-1} \sum_{l \in A_1} (x_l^{M_1} - \theta d_1)$, and that \mathbf{b}_m is a decreasing sequence that converges to 0.

If $\mathbf{y}^1 = M\mathbf{y}^0 + \mathbf{b}_0$, then repeated application of (5.1) yields

$$\begin{aligned} \mathbf{y}^t &= M^t \mathbf{y}^0 + [M^{t-1} \mathbf{b}_0 + M^{t-2} \mathbf{b}_1 + \dots + M \mathbf{b}_{t-2} + \mathbf{b}_{t-1}] \\ &\leq M^t \mathbf{y}^0 + [M^{t-1} + M^{t-2} + \dots + M + I] \mathbf{b}_0. \end{aligned}$$

But M is a right stochastic matrix, and so by the Perron-Frobenius Theorem, it is invertible and has a unique largest eigenvalue of modulus 1. Hence $M^{t-1} + M^{t-2} + \dots + M + I = [1 - M]^{-1} [1 - M^{t+1}]$ is bounded as t increases. As discussed in Section 3, the Perron-Frobenius Theorem also implies that $M^t \mathbf{y}^0$ converges as t increases to the unique fixed point which is the consistent solution to the bankruptcy problem $T(\mathbf{d}|_{N \setminus A_1}, E_{M_1})$ where $E_{M_1} = \sum y_i^0 = E - \sum_{l \in A_1} (x_l^{M_1} - \theta d_1)$. Thus $|\mathbf{y}^t \rightarrow -T(\mathbf{d}|_{N \setminus A_1}, E_{M_1})| \leq C \mathbf{b}_0$ as $t \rightarrow \infty$. Since \mathbf{b}_0 can be made arbitrarily small and $E_{M_1} \rightarrow E$ as M_1 increases (and the TAL rule is continuous in E), this implies x_i^m converges to the TAL solution for all $l \notin A_1$. \square

The proofs of Stage 2 and the remaining Stages continue in this manner.

6. Conclusion and Future Directions

In summary, we have defined a k -averaging dynamic process for consistent bankruptcy rules, and have shown that for rules that are monotonic in the estate size the sequence of allocations converges to the consistent solution. In addition, we have provided a direct proof of this convergence for the Proportional rule, and for the TAL rules when averaging among subsets of size $k = 2$. The latter proof is motivated by the θ -coalitional procedure and provides insight into when the convergence is finite or asymptotic. These ideas extend naturally to a more general topological setting.

By representing each player as a node and placing an edge between each pair of players, the resulting graph describes the network associated with the pairwise

averaging dynamics: this is the complete graph of n vertices, K_n . For averaging over all subsets of a fixed size, a similar hypergraph description applies. A natural question is to consider whether convergence occurs over other networks. For example, under the pairwise averaging dynamics, a ring structure in which player i averages its outcome under a consistent bankruptcy rule between players $i - 1$ and i and players i and $i + 1$ (where player 1 is paired with players 2 and n , and player n is paired with players $n - 1$ and 1) would be a natural network to consider. This network is represented by the n -vertex cycle C_n .

Because the pairwise averaging dynamics considers all pairs of players, the calculations are cumbersome. A network with fewer edges would simplify the calculations. However, the associated dynamic process may not converge as quickly. It would be interesting to compare the effect of the network structure on convergence and the average number of steps it takes a process to get within epsilon of the n -player solution, where the average is over all initial allocations in the simplex. Similarly, the maximum length of time to get close to the n -player solution from an initial allocation in the simplex would also be of interest.

Appendix: Proofs of Lemmas 5.4 and 5.5

PROOF. (of Lemma 5.4) (a) If $d_i \geq d_k$ then $T(i, k, m) \geq (x_i^m + x_k^m)/2 \geq x_i^m$. If $d_i < d_k$ then either $T(i, k, m) = (x_i^m + x_k^m)/2 \geq x_i^m$ or $T(i, k, m) \geq \theta d_i \geq x_i^m$. (b) If $d_j \leq d_k$ then either $T(j, k, m) \leq \theta d_j \leq x_j^m$ or $T(j, k, m) = (x_j^m + x_k^m + d_j - d_k)/2 \leq x_j^m$. If $d_k < d_j$ then this implies $x_k^m \leq x_j^m$. If $x_k^m \leq \theta d_k$ then by (a), $T(k, j, m) \geq x_k^m$ which implies $T(j, k, m) \leq x_j^m$. If $x_k^m > \theta d_k$ then $x_j^m + x_k^m \geq d_j + (2\theta - 1)d_k$ so $T(j, k, m) = (x_j^m + x_k^m + d_j - d_k)/2 \leq x_j^m$. □

PROOF. (of Lemma 5.5) (a) If $d_p \leq d_q$ then $T(p, q, M) \leq (x_p^M + x_q^M)/2 = x_p^M + (x_q^M - x_p^M)/2 \leq x_p^M - \delta/2$. If $d_p > d_q$, then either: (i) $T(p, q, M) = (x_p^M + x_q^M)/2 \leq x_p^M - \delta/2$; (ii) $T(p, q, M) = x_p^M + x_q^M - \theta d_q \leq x_p^M - \delta$; or (iii) $T(p, q, M) = (x_p^M + x_q^M + d_p - d_q)/2 = x_p^M + [d_p - x_p^M + x_q^M - d_q]/2$. But this occurs only if $x_p^M + x_q^M \geq d_p + (2\theta - 1)d_q$ which implies $x_p^M \geq d_p + (2\theta - 1)d_q - \theta d_1 + \delta$. So $[d_p - x_p^M + x_q^M - d_q]/2 \leq \theta d_1 - \theta d_q - \delta \leq -\delta$ and $T(p, q, M) \leq x_p^M - \delta$.

(b) Since $x_p^M + x_q^M \leq 2\theta d_1$, $T(p, q, M) = (x_p^M + x_q^M)/2 = x_p^M + (x_q^M - x_p^M)/2 \geq x_p^M + \lambda/2$.

(c) If $d_p \geq d_q$ then $T(p, q, M) \geq (x_p^M + x_q^M)/2 = x_p^M + (x_q^M - x_p^M)/2 \geq x_p^M + (\delta - \lambda)/2 \geq x_p^M + \delta/4 \geq x_p^M + \beta/2$. If $d_p < d_q$ then either $T(p, q, M) = (x_p^M + x_q^M)/2 \geq x_p^M + \beta/2$ or $T(p, q, M) \geq \theta d_p = x_p^M + \theta d_p - \theta d_1 - \lambda \geq x_p^M + \theta(d_{i_2} - d_1) - \lambda \geq x_p^M + \theta(d_{i_2} - d_1)/2 \geq x_p^M + \beta/2$. □

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MATHEMATICAL REVIEWS, 416 FOURTH STREET, ANN ARBOR, MICHIGAN 48103
E-mail address: maj@ams.org

DEPARTMENT OF NATURAL SCIENCE AND MATHEMATICS, EUGENE LANG COLLEGE THE NEW
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